# Interest Rate Models — Lecture 1 Notes

*Sections 1.1–1.2: Bank account, discounting, zero-coupon bonds, and spot rates*

## 1 The bank account and the short rate

A money-market account is the basic riskless compounding object in the model.

**Definition.** Let $B(t)$ denote the value of the bank account at time $t \geq 0$, with $B(0) = 1$. Its dynamics are

$$dB(t) = r_t B(t)\, dt, \qquad B(0) = 1. \tag{1.1}$$

Here $r_t$ is the *instantaneous spot rate* (or *short rate*).

Separating variables and integrating gives

$$B(t) = \exp\left(\int_0^t r_s\, ds\right). \tag{1.2}$$

Thus the bank account accumulates continuously at the short rate path.

For a small increment $\Delta t$, a first-order expansion gives

$$B(t + \Delta t) \approx B(t)\big(1 + r(t)\Delta t\big), \tag{1.3}$$

so that

$$\frac{B(t + \Delta t) - B(t)}{B(t)} \approx r(t)\Delta t.$$

Hence $r(t)$ is the local proportional growth rate of the bank account.

## 2 Discounting and zero-coupon bonds

To compare cash at different dates, first consider the deterministic case. If $A$ is invested at time 0, then its value at time $t$ is $AB(t)$ and at time $T$ is $AB(T)$. If we want exactly one unit of currency at time $T$, impose

$$AB(T) = 1 \qquad \Longrightarrow \qquad A = \frac{1}{B(T)}.$$

Therefore, the value at time $t$ of one unit paid at time $T$ is

$$AB(t) = \frac{B(t)}{B(T)}.$$

**Definition.** The *discount factor* from $T$ back to $t$ is

$$D(t, T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_s\, ds\right). \tag{1.4}$$

When rates are deterministic, $D(t, T)$ is known at time $t$. When rates are stochastic, $D(t, T)$ is random because it depends on the future path of $r$.

**Definition.** A $T$-maturity *zero-coupon bond* is a contract that pays one unit of currency at time $T$ and nothing before then. Its time-$t$ price is denoted by $P(t, T)$, with

$$P(T, T) = 1.$$

**Key distinction.** If rates are stochastic, the discount factor $D(t, T)$ is random at time $t$, while the bond price $P(t, T)$ must be known at time $t$ because it is the observed price of a traded claim.

**Time to maturity.** The time between $t$ and $T$ is measured by a year fraction $\tau(t, T)$. Different markets may use different day-count conventions.

## 3 Spot rates and compounding conventions

Different spot-rate conventions are different ways of encoding the same bond price $P(t,T)$.

### 3.1 Continuously compounded spot rate

**Definition.** The continuously compounded spot rate is

$$R(t,T) = -\frac{\ln P(t,T)}{\tau(t,T)}. \tag{1.5}$$

Equivalently,

$$e^{R(t,T)\tau(t,T)}P(t,T) = 1, \qquad P(t,T) = e^{-R(t,T)\tau(t,T)}. \tag{1.6}$$

### 3.2 Simply compounded spot rate

**Definition.** The simply compounded spot rate is

$$L(t,T) = \frac{1 - P(t,T)}{\tau(t,T)P(t,T)}. \tag{1.8}$$

Equivalently,

$$P(t,T)(1 + L(t,T)\tau(t,T)) = 1, \qquad P(t,T) = \frac{1}{1 + L(t,T)\tau(t,T)}. \tag{1.9}$$

### 3.3 Annually compounded spot rate

**Definition.** The annually compounded spot rate is

$$Y(t,T) = \frac{1}{[P(t,T)]^{1/\tau(t,T)}} - 1. \tag{1.10}$$

Equivalently,

$$P(t,T)(1 + Y(t,T))^{\tau(t,T)} = 1, \qquad P(t,T) = \frac{1}{(1 + Y(t,T))^{\tau(t,T)}}. \tag{1.11}$$

### 3.4 $k$-times-per-year compounding

**Definition.** If compounding occurs $k$ times per year, then

$$Y^k(t,T) = k\left(\frac{1}{[P(t,T)]^{1/(k\tau(t,T))}} - 1\right). \tag{1.13}$$

Equivalently,

$$P(t,T)\left(1 + \frac{Y^k(t,T)}{k}\right)^{k\tau(t,T)} = 1. \tag{1.14}$$

As $k \to \infty$, discrete compounding converges to continuous compounding:

$$\lim_{k\to\infty}\left(1 + \frac{Y}{k}\right)^{k\tau} = e^{Y\tau}, \qquad \lim_{k\to\infty} Y^k(t,T) = R(t,T).$$

This is why the exponential function appears naturally in continuous compounding.

## Summary

- $B(t)$ is the riskless money-market account; $r_t$ is its instantaneous growth rate.
- $D(t,T)$ discounts one unit at $T$ back to time $t$.
- $P(t,T)$ is the price of a zero-coupon bond paying one unit at $T$.
- $R(t,T)$, $L(t,T)$, $Y(t,T)$, and $Y^k(t,T)$ are different compounding conventions for the same discounting object.

*Next lecture: forward rates — locking in a rate today for borrowing or lending over a future period.*