Interest Rate Models: Theory and Practice

Problem Set 1

Part A

Part A concerns the basic structure of discounting and spot-rate conventions.

Problem A-1. From the bank account to discounting

Let $B(t)$ denote the money-market account, with dynamics

$$dB(t) = r_t B(t)\,dt, \qquad B(0) = 1.$$

(a) Derive the representation

$$B(t) = \exp\left(\int_0^t r_s\,ds\right).$$

(b) In the deterministic-rate case, show that the value at time t of one unit of currency paid at time T is

$$\frac{B(t)}{B(T)}.$$

(c) Deduce the formula

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_s\,ds\right).$$

(d) Explain the financial meaning of r_t, $B(t)$, and $D(t,T)$, and describe how these three objects are related.

Problem A-2. Zero-coupon bonds and spot rates

Let $P(t,T)$ denote the price at time t of a zero-coupon bond maturing at time T, and let $\tau = \tau(t,T)$.

(a) Define the continuously compounded spot rate $R(t,T)$, the simply compounded spot rate $L(t,T)$, and the annually compounded spot rate $Y(t,T)$ in terms of $P(t,T)$.

(b) Solve each definition in part (a) for $P(t,T)$.

(c) Explain why $R(t,T)$, $L(t,T)$, and $Y(t,T)$ should be regarded as different representations of the same underlying discounting object rather than different economic quantities.

Part B

Part B concerns the main conceptual distinctions and limiting arguments from the lecture.

Problem B-1. Discount factor versus bond price

Assume that interest rates are stochastic.

(a) Explain why $D(t,T)$ is generally random at time t.

(b) Explain why $P(t,T)$, as the time-t price of a traded zero-coupon bond maturing at T, must be known at time t.

(c) Explain why one should not identify $D(t,T)$ and $P(t,T)$ pathwise in the stochastic case.

(d) Explain why this distinction disappears in the deterministic-rate case.

This problem should be answered carefully and precisely.

Problem B-2. Discrete versus continuous compounding

The k-times-per-year compounded spot rate is defined by

$$Y^k(t,T) = k\left([P(t,T)]^{-1/(k\tau(t,T))} - 1\right).$$

(a) Show that

$$P(t,T) = \frac{1}{\left(1 + \frac{Y^k(t,T)}{k}\right)^{k\tau(t,T)}}.$$

(b) Show that for fixed y,

$$\lim_{k\to\infty}\left(1 + \frac{y}{k}\right)^k = e^y.$$

(c) Use part (b) to explain why continuous compounding appears as the limit of increasingly frequent compounding.

(d) Explain why the exponential function arises naturally in continuous-time discounting, rather than being introduced *ad hoc*.

Problem B-3. Numerical comparison and structural interpretation

Suppose $\tau(t,T) = 2$ and $P(t,T) = 0.90$.

(a) Compute the continuously compounded spot rate $R(t,T)$, the simply compounded spot rate $L(t,T)$, and the annually compounded spot rate $Y(t,T)$.

(b) The three spot rates in part (a) are numerically different, even though they are derived from the same bond price. Explain why this does *not* represent a contradiction.

In particular, address the following points:

1. What is the single underlying financial object common to all three calculations?
2. Why does the numerical value of a quoted spot rate depend on the compounding convention?
3. Why can comparing quoted rates across conventions without conversion be misleading?

Problem B-4. Short-maturity limit

Using the definitions of $R(t,T)$, $L(t,T)$, and $Y(t,T)$, explain heuristically why all three spot-rate conventions should approach the short rate as $T \to t^+$. A fully rigorous proof is not required, but your argument should clearly explain the limiting idea.