Interest Rate Models: Theory and Practice

Solutions — Problem Set 1

Part A

Part A concerns the basic structure of discounting and spot-rate conventions.

Problem A-1. From the bank account to discounting

Let $B(t)$ denote the money-market account, with dynamics

$$dB(t) = r_t B(t)\, dt, \qquad B(0) = 1.$$

(a) Derive the representation

$$B(t) = \exp\left(\int_0^t r_s\, ds\right).$$

(b) In the deterministic-rate case, show that the value at time t of one unit of currency paid at time T is

$$\frac{B(t)}{B(T)}.$$

(c) Deduce the formula

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_s\, ds\right).$$

(d) Explain the financial meaning of r_t, $B(t)$, and $D(t,T)$, and describe how these three objects are related.

Solution.

(a) Since $B(t) > 0$, we may separate variables:

$$\frac{dB(t)}{B(t)} = r_t\, dt.$$

Integrating from 0 to t gives

$$\int_{B(0)}^{B(t)} \frac{dB}{B} = \int_0^t r_s\, ds.$$

Using $B(0) = 1$,

$$\ln B(t) - \ln 1 = \int_0^t r_s\, ds,$$

hence

$$B(t) = \exp\left(\int_0^t r_s\, ds\right).$$

(b) In the deterministic case, invest an amount A at time 0 so that it grows to one unit at time T:

$$AB(T) = 1 \quad\Longrightarrow\quad A = \frac{1}{B(T)}.$$

Therefore its value at time t is

$$AB(t) = \frac{1}{B(T)} B(t) = \frac{B(t)}{B(T)}.$$

Thus the value at time t of one unit paid at time T is $B(t)/B(T)$.

(c) By definition, the discount factor is this time-t value of one unit paid at time T:

$$D(t,T) = \frac{B(t)}{B(T)}.$$

Using the representation from part (a),

$$\frac{B(t)}{B(T)} = \exp\left(\int_0^t r_s\,ds - \int_0^T r_s\,ds\right) = \exp\left(-\int_t^T r_s\,ds\right).$$

Hence

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_s\,ds\right).$$

(d) One possible answer is as follows. The short rate r_t is the instantaneous proportional growth rate of the money-market account. The process $B(t)$ is the value of the riskless compounding account at time t. The discount factor $D(t,T)$ converts one unit paid at time T into its value at time t. These objects are linked by the fact that the bank account accumulates at rate r_t, while discounting reverses that accumulation:

$$dB(t) = r_t B(t)\,dt, \qquad D(t,T) = \frac{B(t)}{B(T)}.$$

Problem A-2. Zero-coupon bonds and spot rates

Let $P(t,T)$ denote the price at time t of a zero-coupon bond maturing at time T, and let $\tau = \tau(t,T)$.

(a) Define the continuously compounded spot rate $R(t,T)$, the simply compounded spot rate $L(t,T)$, and the annually compounded spot rate $Y(t,T)$ in terms of $P(t,T)$.

(b) Solve each definition in part (a) for $P(t,T)$.

(c) Explain why $R(t,T)$, $L(t,T)$, and $Y(t,T)$ should be regarded as different representations of the same underlying discounting object rather than different economic quantities.

Solution.

(a) The three standard definitions are

$$R(t,T) = -\frac{\ln P(t,T)}{\tau(t,T)},$$

$$L(t,T) = \frac{1 - P(t,T)}{\tau(t,T)P(t,T)},$$

$$Y(t,T) = \frac{1}{[P(t,T)]^{1/\tau(t,T)}} - 1.$$

(b) Solving each for $P(t,T)$ gives

$$P(t,T) = e^{-R(t,T)\tau(t,T)},$$

$$P(t,T) = \frac{1}{1 + L(t,T)\tau(t,T)},$$

$$P(t,T) = \frac{1}{(1 + Y(t,T))^{\tau(t,T)}}.$$

(c) The underlying primitive object is the zero-coupon bond price $P(t,T)$ itself, i.e. the time-t value of receiving one unit at time T. The quantities $R(t,T)$, $L(t,T)$, and $Y(t,T)$ are obtained from this same price by imposing different compounding conventions. Thus they are not distinct economic objects; they are different quoting languages for the same discounting information.

Part B

Part B concerns the main conceptual distinctions and limiting arguments from the lecture.

Problem B-1. Discount factor versus bond price

Assume that interest rates are stochastic.

(a) Explain why $D(t,T)$ is generally random at time t.

(b) Explain why $P(t,T)$, as the time-t price of a traded zero-coupon bond maturing at T, must be known at time t.

(c) Explain why one should not identify $D(t,T)$ and $P(t,T)$ pathwise in the stochastic case.

(d) Explain why this distinction disappears in the deterministic-rate case.

This problem should be answered carefully and precisely.

Solution. One possible answer is as follows.

(a) By definition,

$$D(t,T) = \exp\left(-\int_t^T r_s\,ds\right).$$

At time t, the future short-rate path $\{r_s\}_{s\in[t,T]}$ has not yet been realized. Since the integral depends on that future path, $D(t,T)$ is generally random from the perspective of time t.

(b) The quantity $P(t,T)$ is the market price at time t of a traded zero-coupon bond maturing at T. A market price observed at time t must be known at time t; otherwise it could not serve as the time-t price of the claim.

(c) In the stochastic case, $D(t,T)$ is a future-path-dependent random variable, while $P(t,T)$ is a current market price. Therefore one cannot identify them *pathwise* at time t. The correct relation, introduced later under no-arbitrage, is that the bond price is linked to an appropriate expectation of the discount factor, not to the realized future discount factor itself.

(d) If rates are deterministic, then the future path of r_s over $[t,T]$ is already known at time t. Hence $D(t,T)$ is no longer random, and the time-t value of one unit paid at T is just that deterministic number. In this case the distinction disappears and one has

$$P(t,T) = D(t,T).$$

Problem B-2. Discrete versus continuous compounding

The k-times-per-year compounded spot rate is defined by

$$Y^k(t,T) = k\left([P(t,T)]^{-1/(k\tau(t,T))} - 1\right).$$

(a) Show that

$$P(t,T) = \frac{1}{\left(1 + \frac{Y^k(t,T)}{k}\right)^{k\tau(t,T)}}.$$

(b) Show that for fixed y,

$$\lim_{k\to\infty} \left(1 + \frac{y}{k}\right)^k = e^y.$$

(c) Use part (b) to explain why continuous compounding appears as the limit of increasingly frequent compounding.

(d) Explain why the exponential function arises naturally in continuous-time discounting, rather than being introduced *ad hoc*.

Solution.

(a) Starting from the definition,

$$\frac{Y^k(t,T)}{k} = [P(t,T)]^{-1/(k\tau(t,T))} - 1.$$

Therefore

$$1 + \frac{Y^k(t,T)}{k} = [P(t,T)]^{-1/(k\tau(t,T))}.$$

Raising both sides to the power $k\tau(t,T)$ gives

$$\left(1 + \frac{Y^k(t,T)}{k}\right)^{k\tau(t,T)} = \frac{1}{P(t,T)},$$

so

$$P(t,T) = \frac{1}{\left(1 + \frac{Y^k(t,T)}{k}\right)^{k\tau(t,T)}}.$$

(b) This is the standard exponential limit from calculus:

$$\lim_{k\to\infty} \left(1 + \frac{y}{k}\right)^k = e^y.$$

(c) Under k-times-per-year compounding,

$$P(t,T) = \frac{1}{\left(1 + \frac{Y^k(t,T)}{k}\right)^{k\tau(t,T)}}.$$

As $k \to \infty$, part (b) shows that the discrete compounding term converges to an exponential. Hence the limiting discount formula becomes

$$P(t,T) = e^{-R(t,T)\tau(t,T)}$$

for the corresponding continuously compounded rate. Thus continuous compounding is precisely the limit of increasingly frequent discrete compounding.

(d) The exponential function is not inserted by hand. It arises because continuous-time accumulation is built from repeated local proportional growth over smaller and smaller time intervals. Equivalently, the bank-account differential equation

$$dB(t) = r_t B(t)\, dt$$

has exponential integral solution. Continuous discounting is simply the inverse of that accumulation rule. In this sense the exponential is forced by the structure of continuous-time proportional growth.

Problem B-3. Numerical comparison and structural interpretation

Suppose $\tau(t, T) = 2$ and $P(t, T) = 0.90$.

(a) Compute the continuously compounded spot rate $R(t, T)$, the simply compounded spot rate $L(t, T)$, and the annually compounded spot rate $Y(t, T)$.

(b) The three spot rates in part (a) are numerically different, even though they are derived from the same bond price. Explain why this does not represent a contradiction.

In particular, address the following points:

1. What is the single underlying financial object common to all three calculations?
2. Why does the numerical value of a quoted spot rate depend on the compounding convention?
3. Why can comparing quoted rates across conventions without conversion be misleading?

Solution.

(a) Since $\tau(t, T) = 2$ and $P(t, T) = 0.90$,

$$R(t, T) = -\frac{\ln(0.90)}{2} \approx 0.05268.$$

Hence the continuously compounded spot rate is approximately 5.268%.

For simple compounding,

$$L(t, T) = \frac{1 - 0.90}{2 \cdot 0.90} = \frac{0.10}{1.80} \approx 0.05556,$$

so the simply compounded spot rate is approximately 5.556%.

For annual compounding,

$$Y(t, T) = \frac{1}{(0.90)^{1/2}} - 1 \approx 0.05409,$$

so the annually compounded spot rate is approximately 5.409%.

(b) One possible answer is as follows.

1. The common underlying financial object is the zero-coupon bond price $P(t, T) = 0.90$, i.e. the time-t value of a claim paying one unit at time T.
2. A quoted spot rate is not primitive. It is obtained by translating the same bond price into a particular compounding language. Different compounding conventions imply different algebraic mappings between rates and discount factors, so the numerical value required to encode the same price will generally differ across conventions.
3. Comparing quoted rates across conventions without first converting them to a common basis can be misleading because a larger quoted rate may reflect nothing more than a different representation rule. The economically meaningful comparison is at the level of the implied discount factor or bond price, not at the level of unadjusted quoted rates.

Therefore the difference among the numerical values is not a contradiction. The economic value is the same; only the compounding convention used to report it has changed.

Problem B-4. Short-maturity limit

Using the definitions of $R(t,T)$, $L(t,T)$, and $Y(t,T)$, explain heuristically why all three spot-rate conventions should approach the short rate as $T \to t^+$. A fully rigorous proof is not required, but your argument should clearly explain the limiting idea.

Solution. One possible heuristic argument is as follows. As $T \to t^+$, the maturity interval becomes infinitesimally short, so all sensible compounding conventions must agree to first order. Over a very short interval Δt, the bank account satisfies

$$B(t + \Delta t) \approx B(t)(1 + r(t)\Delta t),$$

which implies that the time-t value of one unit paid at $t + \Delta t$ is approximately

$$P(t, t + \Delta t) \approx \frac{1}{1 + r(t)\Delta t} \approx 1 - r(t)\Delta t.$$

Substituting this first-order form into the definitions of $R(t,T)$, $L(t,T)$, and $Y(t,T)$ shows that each convention extracts the same local rate $r(t)$ in the limit. Thus the short rate is the common instantaneous limit of the various spot-rate conventions.