Interest Rate Models — Lecture 2 Notes

Sections 1.3–1.6: term structure, forward rates, swaps, caps/floors, and swaptions

1 From individual rates to the term structure

Last lecture introduced the objects $P(t,T)$, $R(t,T)$, $L(t,T)$, and $Y(t,T)$ for a single maturity T. The next step is to view these across maturities. The zero-coupon curve at time t is the term structure

$$T \mapsto \begin{cases} L(t,T), & t < T \leq t+1, \\ Y(t,T), & T > t+1. \end{cases} \tag{1.16}$$

Thus the market is not about one isolated spot rate, but about the whole curve of rates across maturity.

2 Forward-rate agreements and forward rates

A forward-rate agreement (FRA) involves current time t, expiry $T > t$, and maturity $S > T$. It locks in a simply compounded rate K for the future interval $[T,S]$. At time S, the contract exchanges a fixed payment against a floating payment based on $L(T,S)$, so its value is

$$N\tau(T,S)\big(K - L(T,S)\big). \tag{1.17}$$

Using the simply compounded spot-rate identity

$$L(T,S) = \frac{1 - P(T,S)}{\tau(T,S)P(T,S)},$$

this can be rewritten as

$$N\left[\tau(T,S)K - \frac{1}{P(T,S)} + 1\right]. \tag{1.18}$$

The term $A = 1/P(T,S)$ is an amount held at time S whose value at time T is exactly 1, since $P(T,S)A = 1$. Discounting the three terms of (**??**) back to time t gives

$$\mathrm{FRA}(t,T,S,\tau(T,S),N,K) = N\big[P(t,S)\tau(T,S)K - P(t,T) + P(t,S)\big]. \tag{1.19}$$

The fair fixed rate is the one that makes the FRA worth 0 at time t.

Definition. The simply compounded forward rate for expiry T and maturity S is

$$F(t;T,S) := \frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right). \tag{1.20}$$

Equivalently, the FRA value may be written as

$$\mathrm{FRA}(t,T,S,\tau(T,S),N,K) = NP(t,S)\tau(T,S)\big(K - F(t;T,S)\big). \tag{1.21}$$

So the forward rate is the unique rate that renders the FRA fair today.

3 Instantaneous forward rates

Forward rates over one interval lead naturally to a forward-rate term structure. Letting the maturity collapse toward the expiry gives

$$\lim_{S \to T^+} F(t;T,S) = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T} = -\frac{\partial \ln P(t,T)}{\partial T}. \tag{1.22}$$

This motivates the instantaneous forward rate

$$f(t,T) := \lim_{S\to T^+} F(t;T,S) = -\frac{\partial \ln P(t,T)}{\partial T}, \qquad P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right). \tag{1.23}$$

Thus $f(t,T)$ plays for forward rates the same structural role that the spot-rate conventions played for single-maturity discounting: once the forward-rate term structure is known, the zero-coupon bond price can be reconstructed.

4 Interest-rate swaps and forward swap rates

A swap can be viewed as a strip of FRAs. For a receiver forward swap (receive fixed, pay floating) with reset dates $T_\alpha, \dots, T_\beta$, year fractions $\tau_{\alpha+1}, \dots, \tau_\beta$, nominal N, and fixed rate K, the value is

$$\begin{aligned} \mathrm{RFS}(t, \mathbf{T}, \boldsymbol{\tau}, N, K) &= \sum_{i=\alpha+1}^{\beta} \mathrm{FRA}(t, T_{i-1}, T_i, \tau_i, N, K) && (1) \\ &= -NP(t,T_\alpha) + NP(t,T_\beta) + N \sum_{i=\alpha+1}^{\beta} \tau_i K P(t,T_i). && (1.24) \end{aligned}$$

The last line makes the structure transparent: the floating leg contributes $-P(t,T_\alpha) + P(t,T_\beta)$, while the fixed leg contributes the coupon-bearing-bond type term $\sum \tau_i K P(t,T_i)$.

The fair swap rate is obtained by setting the receiver swap value to zero. This gives the forward swap rate

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)}. \tag{1.25}$$

The denominator is the annuity (or PVBP), so the forward swap rate is "floating-leg PV divided by annuity."

5 Caps, floors, and Black's formulas

A cap is a strip of caplets, and a floor is a strip of floorlets. Their discounted payoffs are

$$\sum_{i=\alpha+1}^{\beta} D(t,T_i) N \tau_i \big(L(T_{i-1},T_i) - K\big)^+, \qquad \sum_{i=\alpha+1}^{\beta} D(t,T_i) N \tau_i \big(K - L(T_{i-1},T_i)\big)^+.$$

Market practice prices them by Black's formula:

$$\mathrm{Cap}^{\mathrm{Black}}(0, \mathbf{T}, \boldsymbol{\tau}, N, K, \sigma_{\alpha,\beta}) = N \sum_{i=\alpha+1}^{\beta} P(0,T_i)\tau_i \,\mathrm{Bl}\big(K, F(0;T_{i-1},T_i), v_i, 1\big), \tag{1.26}$$

$$\mathrm{Flr}^{\mathrm{Black}}(0, \mathbf{T}, \boldsymbol{\tau}, N, K, \sigma_{\alpha,\beta}) = N \sum_{i=\alpha+1}^{\beta} P(0,T_i)\tau_i \,\mathrm{Bl}\big(K, F(0;T_{i-1},T_i), v_i, -1\big), \tag{1.27}$$

where

$$\mathrm{Bl}(K,F,v,\omega) = F^\omega \Phi(\omega d_1) - K^\omega \Phi(\omega d_2), \quad d_1 = \frac{\ln(F/K) + v^2/2}{v}, \quad d_2 = \frac{\ln(F/K) - v^2/2}{v}, \quad v_i = \sigma_{\alpha,\beta}\sqrt{T_{i-1}}.$$

A cap protects against rising rates; a floor protects against falling rates. Also,

$$(L-K)^+ - (K-L)^+ = L - K,$$

so caps, floors, and swaps are linked by cap–floor parity.

6 Swaptions

A swaption is an option to enter a swap. A payer swaption gives the right to pay fixed; a receiver swaption gives the right to receive fixed. Unlike a cap, a swaption depends on the joint behaviour of future rates through the future swap rate. Market practice values payer and receiver swaptions via Black-like formulas:

$$\mathrm{PS}^{\mathrm{Black}}(0, \mathbf{T}, \boldsymbol{\tau}, N, K, \sigma_{\alpha,\beta}) = N\,\mathrm{Bl}\big(K, S_{\alpha,\beta}(0), \sigma_{\alpha,\beta}\sqrt{T_\alpha}, 1\big) \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i), \tag{1.28}$$

$$\mathrm{RS}^{\mathrm{Black}}(0, \mathbf{T}, \boldsymbol{\tau}, N, K, \sigma_{\alpha,\beta}) = N\,\mathrm{Bl}\big(K, S_{\alpha,\beta}(0), \sigma_{\alpha,\beta}\sqrt{T_\alpha}, -1\big) \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i). \tag{1.29}$$

A swaption is ATM when $K = S_{\alpha,\beta}(0)$. Since a cap is the sum of positive caplet payoffs, while a swaption exercises only if the whole swap has positive value, the swaption payoff is bounded above by the corresponding cap-style sum. This is where terminal correlation becomes important.

Summary

- The zero-coupon curve is the term structure of rates across maturities.
- An FRA leads to the forward rate $F(t; T, S)$, and the short-maturity limit gives the instantaneous forward rate $f(t, T)$.
- A swap is a strip of FRAs; the fair fixed rate is the forward swap rate $S_{\alpha,\beta}(t)$.
- Caps/floors and swaptions are option-like interest-rate derivatives, valued in market practice by Black-type formulas.

Next lecture: from static term-structure formulas to no-arbitrage pricing and numeraires.