Interest Rate Models: Theory and Practice
Problem Set 2

Part A

Part A concerns the basic structure of forward rates, swaps, and their relation to bond prices.

Problem A-1. From an FRA payoff to the forward rate

Consider a simply compounded FRA with current time t, expiry $T > t$, maturity $S > T$, nominal N, accrual fraction $\tau(T,S)$, and fixed rate K.

(a) Starting from the time-S payoff $N\tau(T,S)(K - L(T,S))$, show that it can be written in the bond-price form

$$N\left[\tau(T,S)K - \frac{1}{P(T,S)} + 1\right].$$

(b) Discount this payoff back to time t and derive

$$\mathrm{FRA}(t,T,S,\tau(T,S),N,K) = N\big[P(t,S)\tau(T,S)K - P(t,T) + P(t,S)\big].$$

(c) Solve for the unique value of K that makes the FRA fair at time t, and hence derive the forward rate

$$F(t;T,S) = \frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right).$$

Problem A-2. Instantaneous forward rates

(a) Starting from the simply compounded forward rate $F(t;T,S)$, take the limit as $S \to T^+$ and show heuristically that

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}.$$

(b) Deduce that

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right).$$

(c) Explain why the instantaneous forward rate may be viewed as a more granular term-structure object than a spot rate.

Problem A-3. Swaps and the forward swap rate

(a) Show that the value of a receiver forward swap can be written as

$$\mathrm{RFS}(t,\mathbf{T},\boldsymbol{\tau},N,K) = -NP(t,T_\alpha) + NP(t,T_\beta) + N\sum_{i=\alpha+1}^{\beta}\tau_i K P(t,T_i).$$

(b) By imposing fairness at time t, derive the forward swap rate

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\tau_i P(t,T_i)}.$$

(c) Explain carefully why the denominator is naturally interpreted as an annuity (or PVBP).

Part B

Part B concerns the main structural relationships among FRAs, forward curves, caps/floors, and swaptions.

Problem B-1. Reconstructing the term structure from instantaneous forwards

Suppose the instantaneous forward curve $u \mapsto f(t,u)$ is known for all $u \in [t,T]$.

(a) Explain why this information determines $P(t,T)$.

(b) Explain how knowledge of $P(t,T)$ for all maturities then determines simply compounded forward rates and forward swap rates.

(c) In a few precise sentences, compare the roles of the zero-coupon curve and the instantaneous forward curve as representations of the term structure.

Problem B-2. Cap–floor parity and the swap

(a) Show that for any real number x and strike K,

$$(x-K)^+ - (K-x)^+ = x - K.$$

(b) Apply part (a) with $x = L(T_{i-1}, T_i)$ to show that the difference between a caplet and the corresponding floorlet payoff equals the payoff of a single fixed-versus-floating exchange over that period.

(c) Deduce heuristically why "cap − floor = swap" is the natural portfolio identity in this setting.

Problem B-3. Swaptions versus caps

A cap is the sum of individually exercised caplets, whereas a swaption is exercised only if the value of the entire underlying swap is positive at exercise.

(a) Explain why this means the swaption payoff depends on the joint behaviour of future rates rather than only on each rate separately.

(b) Explain heuristically why the payer swaption payoff is bounded above by the corresponding cap-style sum of positive individual terms.

(c) Explain why terminal correlation matters for swaptions in a way that is absent from a simple sum of independent positive-part terms.

Problem B-4. Market formulas and ATM strikes

(a) State the Black-style market formulas for a cap, a floor, a payer swaption, and a receiver swaption.

(b) Explain why a cap/floor is ATM when its strike equals the relevant current forward rate, and why a swaption is ATM when its strike equals $S_{\alpha,\beta}(0)$.

(c) Explain why the volatility parameter entering the swaption formula should not simply be identified with the cap/floor volatility parameter, even though both formulas are Black-like.