Interest Rate Models: Theory and Practice

Solutions to Problem Set 2

Problem A-1. From an FRA payoff to the forward rate

Consider a simply compounded FRA with current time t, expiry $T > t$, maturity $S > T$, nominal N, accrual fraction $\tau(T,S)$, and fixed rate K.

(a) Starting from the time-S payoff $N\tau(T,S)(K - L(T,S))$, show that it can be written in the bond-price form

$$N\left[\tau(T,S)K - \frac{1}{P(T,S)} + 1\right].$$

One has

$$L(T,S) = \frac{1 - P(T,S)}{\tau(T,S)P(T,S)}.$$

Substituting into $N\tau(T,S)(K - L(T,S))$ gives

$$\begin{aligned} N\tau(T,S)\left(K - \frac{1-P(T,S)}{\tau(T,S)P(T,S)}\right) &= N\left[\tau(T,S)K - \frac{1-P(T,S)}{P(T,S)}\right] \\ &= N\left[\tau(T,S)K - \frac{1}{P(T,S)} + 1\right]. \end{aligned}$$

(b) Discount this payoff back to time t and derive

$$\mathrm{FRA}(t,T,S,\tau(T,S),N,K) = N\big[P(t,S)\tau(T,S)K - P(t,T) + P(t,S)\big].$$

Write the payoff at time S as the sum of three terms. The amount $1/P(T,S)$ at time S is worth 1 at time T, hence it is worth $P(t,T)$ at time t. The term $\tau(T,S)K + 1$ at time S is worth

$$P(t,S)\tau(T,S)K + P(t,S)$$

at time t. Therefore,

$$\mathrm{FRA}(t,T,S,\tau(T,S),N,K) = N\big[P(t,S)\tau(T,S)K - P(t,T) + P(t,S)\big].$$

(c) Solve for the unique value of K that makes the FRA fair at time t, and hence derive the forward rate

$$F(t;T,S) = \frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right).$$

Setting the FRA value equal to 0 gives

$$P(t,S)\tau(T,S)K - P(t,T) + P(t,S) = 0.$$

Hence

$$P(t,S)\tau(T,S)K = P(t,T) - P(t,S),$$

and so

$$K = \frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right).$$

This is precisely the simply compounded forward rate $F(t;T,S)$.

Problem A-2. Instantaneous forward rates

(a) Starting from the simply compounded forward rate $F(t;T,S)$, take the limit as $S \to T^+$ and show heuristically that

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}.$$

Starting from

$$F(t;T,S) = \frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right),$$

and using $\tau(T,S) \approx S - T$ when S is very close to T, one gets

$$\begin{aligned}\lim_{S\to T^+} F(t;T,S) &= \lim_{S\to T^+} \frac{1}{S-T}\left(\frac{P(t,T)-P(t,S)}{P(t,S)}\right)\\ &= -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T} = -\frac{\partial \ln P(t,T)}{\partial T}.\end{aligned}$$

This limit defines $f(t,T)$.

(b) Deduce that

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right).$$

Since

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T},$$

integrating from t to T yields

$$\ln P(t,T) - \ln P(t,t) = -\int_t^T f(t,u)\,du.$$

Because $P(t,t) = 1$, we have $\ln P(t,t) = 0$, so

$$\ln P(t,T) = -\int_t^T f(t,u)\,du,$$

which implies

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right).$$

(c) Explain why the instantaneous forward rate may be viewed as a more granular term-structure object than a spot rate.

One possible answer is as follows. A spot rate summarizes discounting from t to a single maturity T. By contrast, $f(t,T)$ describes the local forward rate attached to an infinitesimal interval beginning at T. Therefore the curve $T \mapsto f(t,T)$ resolves the term structure at a finer maturity-by-maturity level.

Problem A-3. Swaps and the forward swap rate

(a) Show that the value of a receiver forward swap can be written as

$$\mathrm{RFS}(t,\mathbf{T},\boldsymbol{\tau},N,K) = -NP(t,T_\alpha) + NP(t,T_\beta) + N\sum_{i=\alpha+1}^{\beta} \tau_i K P(t,T_i).$$

A receiver swap is a strip of FRAs with common fixed rate K. Summing the FRA values over $i = \alpha+1, \dots, \beta$ gives

$$\mathrm{RFS}(t, \mathbf{T}, \boldsymbol{\tau}, N, K) = \sum_{i=\alpha+1}^{\beta} \mathrm{FRA}(t, T_{i-1}, T_i, \tau_i, N, K).$$

Using the bond-price form of each FRA and telescoping the floating-leg terms yields

$$\mathrm{RFS}(t, \mathbf{T}, \boldsymbol{\tau}, N, K) = -NP(t, T_\alpha) + NP(t, T_\beta) + N \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i).$$

(b) By imposing fairness at time t, derive the forward swap rate

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}.$$

Set $\mathrm{RFS}(t, \mathbf{T}, \boldsymbol{\tau}, N, K) = 0$. Then

$$-P(t, T_\alpha) + P(t, T_\beta) + \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) = 0.$$

Solving for K gives

$$K = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}.$$

This fair fixed rate is $S_{\alpha,\beta}(t)$.

(c) Explain carefully why the denominator is naturally interpreted as an annuity (or PVBP).

The fixed leg pays coupons $N\tau_i K$ at the dates T_i. Discounting and summing gives

$$NK \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i).$$

Thus the entire fixed-leg PV is "rate × present value of one basis-unit stream." The denominator is therefore the present value of one unit of fixed coupon paid according to the swap schedule, i.e. the annuity or PVBP.

Problem B-1. Reconstructing the term structure from instantaneous forwards

(a) Explain why this information determines $P(t, T)$.

Since

$$P(t, T) = \exp\left(-\int_t^T f(t, u)\, du\right),$$

knowing $f(t, u)$ for all $u \in [t, T]$ determines the integral and hence determines $P(t, T)$.

(b) Explain how knowledge of $P(t, T)$ for all maturities then determines simply compounded forward rates and forward swap rates.

Once bond prices are known, simply compounded forward rates follow from

$$F(t; T, S) = \frac{1}{\tau(T, S)} \left(\frac{P(t, T)}{P(t, S)} - 1\right),$$

and forward swap rates follow from

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)}.$$

So bond prices are sufficient to recover both families of forward-looking rates.

(c) In a few precise sentences, compare the roles of the zero-coupon curve and the instantaneous forward curve as representations of the term structure.

One possible answer is as follows. The zero-coupon curve records rates across maturities directly in quoted form. The instantaneous forward curve records the local forward-rate density that, once integrated, reconstructs zero-coupon bond prices. Both represent the same term structure, but the forward curve is the more differential or local representation.

Problem B-2. Cap–floor parity and the swap

(a) Show that for any real number x and strike K,

$$(x-K)^+ - (K-x)^+ = x - K.$$

If $x \geq K$, then $(x-K)^+ = x-K$ and $(K-x)^+ = 0$, so the left-hand side equals $x-K$. If $x < K$, then $(x-K)^+ = 0$ and $(K-x)^+ = K-x$, so the left-hand side equals $-(K-x) = x-K$. Thus the identity holds in all cases.

(b) Apply part (a) with $x = L(T_{i-1}, T_i)$ to show that the difference between a caplet and the corresponding floorlet payoff equals the payoff of a single fixed-versus-floating exchange over that period.

Substitute $x = L(T_{i-1}, T_i)$. Then

$$\left(L(T_{i-1},T_i) - K\right)^+ - \left(K - L(T_{i-1},T_i)\right)^+ = L(T_{i-1},T_i) - K.$$

Multiplying by $N\tau_i$ gives exactly the net fixed-versus-floating payment for that accrual period.

(c) Deduce heuristically why "cap − floor = swap" is the natural portfolio identity in this setting.

Summing the caplet-minus-floorlet identity over all payment dates yields the sum of the corresponding linear fixed-versus-floating exchanges. But this sum is precisely the payoff structure of a swap. Therefore cap minus floor reproduces the swap, which is the interest-rate version of call–put parity.

Problem B-3. Swaptions versus caps

(a) Explain why this means the swaption payoff depends on the joint behaviour of future rates rather than only on each rate separately.

A cap decomposes into separate caplets, each exercised period by period. A swaption instead gives one exercise decision on the whole underlying swap at time T_α. Since the future swap rate is built from many future rates jointly, the swaption payoff depends on their combined configuration, not on each one independently.

(b) Explain heuristically why the payer swaption payoff is bounded above by the corresponding cap-style sum of positive individual terms.

The positive part of a sum is never larger than the sum of the positive parts:

$$\left(\sum_i a_i\right)^+ \leq \sum_i a_i^+.$$

A payer swaption takes the positive part after aggregating the whole swap value. A cap takes positive parts term by term and then sums them. Hence the cap-style payoff dominates the swaption payoff.

(c) Explain why terminal correlation matters for swaptions in a way that is absent from a simple sum of independent positive-part terms.

The terminal swap rate depends on how the future rates move together at exercise. Correlation affects the distribution of the whole sum and therefore the probability and size of positive exercise value. For a caplet strip priced period by period, each positive part is attached to its own rate separately, so the central structural issue is much less about the joint terminal configuration and much more about each individual period.

Problem B-4. Market formulas and ATM strikes

(a) State the Black-style market formulas for a cap, a floor, a payer swaption, and a receiver swaption.

The formulas are

$$\mathrm{Cap}^{\mathrm{Black}}(0,\mathbf{T},\boldsymbol{\tau},N,K,\sigma_{\alpha,\beta}) = N\sum_{i=\alpha+1}^{\beta} P(0,T_i)\tau_i\,\mathrm{Bl}\big(K,F(0;T_{i-1},T_i),v_i,1\big),$$

$$\mathrm{Flr}^{\mathrm{Black}}(0,\mathbf{T},\boldsymbol{\tau},N,K,\sigma_{\alpha,\beta}) = N\sum_{i=\alpha+1}^{\beta} P(0,T_i)\tau_i\,\mathrm{Bl}\big(K,F(0;T_{i-1},T_i),v_i,-1\big),$$

$$\mathrm{PS}^{\mathrm{Black}}(0,\mathbf{T},\boldsymbol{\tau},N,K,\sigma_{\alpha,\beta}) = N\,\mathrm{Bl}(K,S_{\alpha,\beta}(0),\sigma_{\alpha,\beta}\sqrt{T_\alpha},1)\sum_{i=\alpha+1}^{\beta}\tau_i P(0,T_i),$$

$$\mathrm{RS}^{\mathrm{Black}}(0,\mathbf{T},\boldsymbol{\tau},N,K,\sigma_{\alpha,\beta}) = N\,\mathrm{Bl}(K,S_{\alpha,\beta}(0),\sigma_{\alpha,\beta}\sqrt{T_\alpha},-1)\sum_{i=\alpha+1}^{\beta}\tau_i P(0,T_i).$$

(b) Explain why a cap/floor is ATM when its strike equals the relevant current forward rate, and why a swaption is ATM when its strike equals $S_{\alpha,\beta}(0)$.

One possible answer is as follows. A caplet or floorlet is centered around the current forward LIBOR for its underlying period, so the natural zero-intrinsic benchmark occurs when strike equals that forward rate. A swaption is an option on the underlying forward swap rate, so its natural ATM strike is the current forward swap rate $S_{\alpha,\beta}(0)$.

(c) Explain why the volatility parameter entering the swaption formula should not simply be identified with the cap/floor volatility parameter, even though both formulas are Black-like.

The cap/floor formula is written on forward LIBOR-type quantities period by period, whereas the swaption formula is written on the forward swap rate, which aggregates many future rates and is affected by their joint movement. Therefore the relevant uncertainty object is different. The formulas have the same Black-like shape, but the underlying rate and its effective volatility are not the same object.