Interest Rate Models — Lecture 3 Notes

Sections 2.1–2.2 and 2.3 through Proposition 2.3.1: no-arbitrage, numeraires, and the first measure-change toolkit

1 No-arbitrage, trading strategies, and self-financing

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}_0)$ over a horizon $T > 0$. The economy contains $K+1$ traded non-dividend-paying securities with price process

$$S = \{S_t : 0 \le t \le T\}, \qquad S_t = (S_t^0, S_t^1, \dots, S_t^K),$$

where S^0 is the bank account. Thus S_t is the vector of all asset prices at time t, while S_t^k is the price of the k-th asset at time t.

Definition. A trading strategy is a predictable process $\phi = \{\phi_t : 0 \le t \le T\}$ with components $\phi^0, \phi^1, \dots, \phi^K$. Its value process is

$$V_t(\phi) = \phi_t S_t = \sum_{k=0}^{K} \phi_t^k S_t^k,$$

and its gains process is

$$G_t(\phi) = \int_0^t \phi_u \, dS_u = \sum_{k=0}^{K} \int_0^t \phi_u^k \, dS_u^k.$$

The component ϕ_t^k is the number of units of asset k held at time t.

Definition. A trading strategy ϕ is *self-financing* if $V(\phi) \ge 0$ and

$$V_t(\phi) = V_0(\phi) + G_t(\phi), \qquad 0 \le t < T. \tag{2.1}$$

This means the portfolio evolves only through gains and losses on the assets already held: one may rebalance, but no outside cash is injected after time 0.

Proposition. A strategy ϕ is self-financing if and only if its discounted value satisfies

$$D(0,t)V_t(\phi) = V_0(\phi) + \int_0^t \phi_u \, d\big(D(0,u)S_u\big).$$

So the self-financing property is preserved after expressing all prices in units of the bank account.

2 Martingale measures and contingent claims

The key link between no-arbitrage and pricing is the existence of a probability measure under which discounted prices are martingales.

Definition. An equivalent martingale measure $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$ such that:

- $\mathbb{Q}_0$ and $\mathbb{Q}$ are equivalent;
- the Radon–Nikodym derivative $d\mathbb{Q}/d\mathbb{Q}_0$ is square-integrable;
- the discounted asset-price process $D(0,\cdot)S$ is an $(\mathbb{F}, \mathbb{Q})$-martingale.

Thus, under $\mathbb{Q}$, the discounted prices have zero drift. Economically, this is the pricing measure under which valuation becomes expectation of discounted future payoffs.

An *arbitrage opportunity* is a self-financing strategy ϕ with $V_0(\phi) = 0$ and

$$\mathbb{Q}_0\{V_T(\phi) > 0\} > 0.$$

If an equivalent martingale measure exists, such opportunities are ruled out.

Definition. A contingent claim is a positive square-integrable random variable H. It is *attainable* if there exists a self-financing strategy ϕ such that

$$V_T(\phi) = H.$$

In that case, ϕ generates the claim, and its time-t price is $\pi_t = V_t(\phi)$.

Proposition. If an equivalent martingale measure $\mathbb{Q}$ exists and H is attainable, then the associated price process is unique and satisfies

$$\pi_t = \mathbb{E}\big(D(t,T)H \mid \mathcal{F}_t\big). \tag{2.2}$$

This is the fundamental no-arbitrage pricing formula.

Definition. A market is *complete* if every contingent claim is attainable.

Hence:

- arbitrage-free market $\iff$ existence of a martingale measure;
- complete market $\iff$ uniqueness of that martingale measure;
- attainable claims have unique no-arbitrage prices.

3 Numeraires and the change of measure

Formula (2.2) is correct, but not always convenient. Under stochastic rates, discounting by $D(t,T)$ may be awkward, especially when the payoff is naturally linked to another traded asset. This motivates the choice of a different numeraire.

Definition. A *numeraire* is any positive non-dividend-paying asset.

A numeraire is the reference asset in whose units all other prices are measured. If Z is chosen as numeraire, then one studies relative prices X/Z rather than X itself. The point is not to change the price of a claim, but to change the measure under which the normalized object becomes a martingale.

If N is a numeraire with associated measure $\mathbb{Q}^N$, then any traded asset X satisfies

$$\frac{X_t}{N_t} = \mathbb{E}_t^N\left[\frac{X_T}{N_T}\right], \qquad 0 \le t \le T. \tag{2.3}$$

Now let U be another numeraire. Then there exists a measure $\mathbb{Q}^U$ such that any attainable claim Y satisfies

$$\frac{Y_t}{U_t} = \mathbb{E}_t^U\left[\frac{Y_T}{U_T}\right], \qquad 0 \le t \le T. \tag{2.4}$$

The corresponding Radon–Nikodym derivative is

$$\frac{d\mathbb{Q}^U}{d\mathbb{Q}^N} = \frac{U_T N_0}{U_0 N_T}. \tag{2.5}$$

The key identity behind the numeraire change is

$$\mathbb{E}^N\left[\frac{Z_T}{N_T}\right] = \mathbb{E}^U\left[\frac{U_0 Z_T}{N_0 U_T}\right], \tag{2.6}$$

valid for any tradable asset price Z. So the pricing formula changes form, but not value.

4 The first numeraire-change toolkit

Three practical facts summarize the technique.

Fact 1. The price of any asset divided by the chosen numeraire is a martingale under the measure associated with that numeraire.

Examples:

- $S_t/B(t)$ is a martingale under the bank-account measure (the risk-neutral measure);
- the forward LIBOR rate is a martingale under the bond numeraire $P(\cdot, T_2)$;
- the forward swap rate is a martingale under the swap-annuity numeraire $\sum_{i=\alpha+1}^{\beta}(T_i - T_{i-1})P(t, T_i)$.

The point is that one chooses the numeraire so that the natural underlying of the product has zero drift.

Fact 2. The claim price is invariant under a numeraire change:

$$\text{Price}_t = \mathbb{E}_t^B\left[\frac{\text{Payoff}(T)}{B(T)/B(t)}\right] = \mathbb{E}_t^S\left[\frac{S_t}{S_T}\,\text{Payoff}(T)\right].$$

Changing numeraire changes the probability viewpoint and the discounting object, but not the actual price.

Fact 3. When the numeraire changes, the Brownian shock is no longer centered in the correct way for the new measure. Girsanov's theorem re-centers the noise, and this recentering appears as a correction to the drift.

Formally, let S be a numeraire with associated measure $\mathbb{Q}^S$, and let an n-vector diffusion X satisfy

$$dX_t = \mu_t^S(X_t)\,dt + \sigma_t(X_t)C\,dW_t^S, \qquad \mathbb{Q}^S,$$

where σ_t is diagonal, W^S is an n-dimensional Brownian motion, and $CC' = \rho$. Under a new numeraire U with measure $\mathbb{Q}^U$, write

$$dX_t = \mu_t^U(X_t)\,dt + \sigma_t(X_t)C\,dW_t^U, \qquad \mathbb{Q}^U.$$

If

$$\alpha_t := \big(\mu_t^S(X_t) - \mu_t^U(X_t)\big)(\sigma_t(X_t)C)^{-1},$$

then the exponential-martingale density process satisfies

$$d\zeta_t = \alpha_t \zeta_t\,dW_t^U. \tag{2.7}$$

Using the Radon–Nikodym relation,

$$\zeta_T = \left.\frac{d\mathbb{Q}^S}{d\mathbb{Q}^U}\right|_{\mathcal{F}_T} = \frac{U_0 S_T}{S_0 U_T}, \tag{2.8}$$

and since ζ is a $\mathbb{Q}^U$-martingale,

$$\zeta_t = \mathbb{E}_t^U(\zeta_T) = \frac{U_0 S_t}{S_0 U_t}. \tag{2.9}$$

Differentiating gives

$$d\zeta_t = \frac{U_0}{S_0}\, d\left(\frac{S_t}{U_t}\right) = \frac{U_0}{S_0}\, \sigma_t^{S/U} C\, dW_t^U, \tag{2.10}$$

where S/U is a martingale under $\mathbb{Q}^U$.

Comparing (2.7) and (2.10) yields the fundamental drift-change formula

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \sigma_t(X_t)\rho\,(\sigma_t^{S/U})'. \tag{2.11}$$

So changing numeraire means adjusting the drift by the amount needed to restore the martingale property under the new measure.

Proposition 2.3.1. If the two numeraires S and U evolve under $\mathbb{Q}^U$ as

$$dS_t = (\cdots)dt + \sigma_t^S C\, dW_t^U, \qquad dU_t = (\cdots)dt + \sigma_t^U C\, dW_t^U,$$

then the drift of X under the numeraire U is

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \sigma_t(X_t)\rho\left(\frac{\sigma_t^S}{S_t} - \frac{\sigma_t^U}{U_t}\right)'. \tag{2.12}$$

This is the practical form of the toolkit: once the original drift, asset volatility, numeraire volatilities, and correlations are known, the new drift follows immediately.

Summary

- No-arbitrage pricing begins with self-financing strategies, martingale measures, and attainable contingent claims.
- A numeraire is the reference asset in whose units prices are expressed.
- Under the measure associated with the chosen numeraire, normalized prices are martingales.
- A change of numeraire leaves prices unchanged, but changes the measure and therefore the drift.
- The drift-correction formulas (2.11)–(2.12) are the first practical toolkit for moving between pricing measures.

Next lecture: continue the numeraire toolkit, introduce the forward measure explicitly, and derive the fundamental pricing formulas.