Interest Rate Models: Theory and Practice

Problem Set 3

Part A

Part A concerns no-arbitrage pricing, self-financing strategies, and the first change-of-numeraire identities.

Problem A-1. Self-financing and discounted values

Let $S = (S^0, S^1, \ldots, S^K)$ be the vector of traded asset prices, where $S_t^0 = B(t)$ is the bank account, and let $\phi = (\phi^0, \phi^1, \ldots, \phi^K)$ be a trading strategy with value process

$$V_t(\phi) = \phi_t S_t = \sum_{k=0}^{K} \phi_t^k S_t^k$$

and gains process

$$G_t(\phi) = \int_0^t \phi_u \, \mathrm{d}S_u = \sum_{k=0}^{K} \int_0^t \phi_u^k \, \mathrm{d}S_u^k.$$

A strategy is self-financing if

$$V_t(\phi) = V_0(\phi) + G_t(\phi), \qquad 0 \le t < T. \tag{2.1}$$

(a) Explain carefully, in financial terms, what the self-financing condition means.

(b) Show that if ϕ is self-financing, then the discounted value process satisfies

$$D(0,t)V_t(\phi) = V_0(\phi) + \int_0^t \phi_u \, \mathrm{d}\big(D(0,u)S_u\big).$$

(c) Explain why this discounted form is conceptually important for no-arbitrage pricing.

Problem A-2. Equivalent martingale measures and attainable claims

Suppose $\mathbb{Q}$ is an equivalent martingale measure, so that the discounted asset-price process $D(0,\cdot)S$ is a martingale under $\mathbb{Q}$. Let H be an attainable contingent claim.

(a) State the pricing formula for the unique no-arbitrage price process π_t associated with H.

(b) Take the particular claim $H \equiv 1$ paid at time T. Show that the pricing formula reduces to

$$\pi_t = P(t,T).$$

(c) Explain why attainability is the key hypothesis behind uniqueness of price here.

Problem A-3. Numeraires and the Radon–Nikodym derivative

Let N be a numeraire with associated measure $\mathbb{Q}^N$, and let U be another numeraire with associated measure $\mathbb{Q}^U$. Proposition 2.2.1 states that for any attainable claim Y,

$$\frac{Y_t}{U_t} = \mathbb{E}_t^U\left[\frac{Y_T}{U_T}\right],$$

and that the Radon–Nikodym derivative is

$$\frac{d\mathbb{Q}^U}{d\mathbb{Q}^N} = \frac{U_T N_0}{U_0 N_T}. \tag{2.5}$$

(a) Explain what a numeraire is, and why changing numeraires may simplify pricing.

(b) Starting from the identity

$$\mathbb{E}^N\left[\frac{Z_T}{N_T}\right] = \mathbb{E}^U\left[\frac{U_0 Z_T}{N_0 U_T}\right], \tag{2.6}$$

explain how formula (2.5) arises.

(c) Explain, in words, why changing the numeraire changes the measure but not the price.

Part B

Part B concerns the numeraire toolkit, martingale examples, and the drift correction formula.

Problem B-1. Fact One and the natural martingale examples

(a) Explain why an asset price divided by its chosen numeraire should have zero drift under the measure associated with that numeraire.

(b) Use this principle to explain why the forward LIBOR rate

$$F_2(t) = \frac{(P(t,T_1) - P(t,T_2))/(T_2 - T_1)}{P(t,T_2)}$$

is a martingale under the T_2-forward measure.

(c) Use the same principle to explain why the forward swap rate

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}(T_i - T_{i-1})P(t,T_i)}$$

is a martingale under the swap measure associated with the annuity

$$\sum_{i=\alpha+1}^{\beta}(T_i - T_{i-1})P(t,T_i).$$

Problem B-2. Fact Two and numeraire invariance of price

Fact Two states that

$$\text{Price}_t = \mathbb{E}_t^B\left[\frac{B(t)}{B(T)}\,\text{Payoff}(T)\right] = \mathbb{E}_t^S\left[\frac{S_t}{S_T}\,\text{Payoff}(T)\right].$$

(a) Explain why this identity is the right formal expression of “same claim, same price.”

(b) In the special case $\text{Payoff}(T) = 1$, explain how the formula reproduces the price of a zero-coupon bond under any convenient numeraire.

(c) Explain why the bank-account numeraire is correct but often inconvenient in interest-rate problems.

Problem B-3. From Girsanov to the drift change formula

Let X be an n-vector diffusion. Under the measure associated with numeraire S, suppose

$$\mathrm{d}X_t = \mu_t^S(X_t)\,\mathrm{d}t + \sigma_t(X_t)C\,\mathrm{d}W_t^S,$$

and under the measure associated with numeraire U suppose

$$\mathrm{d}X_t = \mu_t^U(X_t)\,\mathrm{d}t + \sigma_t(X_t)C\,\mathrm{d}W_t^U.$$

Assume further that under $\mathbb{Q}^U$,

$$\mathrm{d}S_t = (\cdots)\mathrm{d}t + \sigma_t^S C\,\mathrm{d}W_t^U, \qquad \mathrm{d}U_t = (\cdots)\mathrm{d}t + \sigma_t^U C\,\mathrm{d}W_t^U.$$

(a) Starting from

$$\zeta_t = \frac{U_0 S_t}{S_0 U_t}, \tag{2.9}$$

explain why ζ is the density process relating $\mathbb{Q}^S$ and $\mathbb{Q}^U$.

(b) Use the martingale dynamics of S_t/U_t under $\mathbb{Q}^U$ to derive

$$\mathrm{d}\zeta_t = \frac{U_0}{S_0}\,\frac{S_t}{U_t}\,\sigma_t^{S/U} C\,\mathrm{d}W_t^U. \tag{2.10}$$

(c) By comparing this with the exponential-martingale dynamics

$$\mathrm{d}\zeta_t = \alpha_t \zeta_t\,\mathrm{d}W_t^U, \tag{2.7}$$

derive

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \sigma_t(X_t)\,\rho\left(\frac{\sigma_t^S}{S_t} - \frac{\sigma_t^U}{U_t}\right)', \tag{2.12}$$

where $\rho = CC'$. You may quote formula (2.11) as an intermediate step.

Problem B-4. Brownian recentering and economic interpretation

This problem is conceptual rather than computational.

(a) Explain why a Brownian motion that is driftless under one numeraire-associated measure need not remain driftless after the numeraire is changed.

(b) Explain, in words, what is meant by "re-centering the Brownian noise."

(c) Explain why this re-centering shows up in practice as a correction to the drift term of the asset dynamics.

(d) In a few precise sentences, summarize the economic meaning of formula (2.12).