Interest Rate Models: Theory and Practice

Solutions to Problem Set 3

Problem A-1. Self-financing and discounted values

Let $S = (S^0, S^1, \dots, S^K)$ be the vector of traded asset prices, where $S_t^0 = B(t)$ is the bank account, and let $\phi = (\phi^0, \phi^1, \dots, \phi^K)$ be a trading strategy with value process

$$V_t(\phi) = \phi_t S_t = \sum_{k=0}^{K} \phi_t^k S_t^k$$

and gains process

$$G_t(\phi) = \int_0^t \phi_u \, \mathrm{d}S_u = \sum_{k=0}^{K} \int_0^t \phi_u^k \, \mathrm{d}S_u^k.$$

A strategy is self-financing if

$$V_t(\phi) = V_0(\phi) + G_t(\phi), \qquad 0 \le t < T. \tag{2.1}$$

(a) Explain carefully, in financial terms, what the self-financing condition means.

A self-financing strategy is one whose value changes only because the assets already held in the portfolio move in price. The investor is allowed to rebalance across assets, but is not allowed to inject new cash into the portfolio or withdraw cash from it after time 0. Thus all future gains and losses are generated internally by trading the existing holdings.

(b) Show that if ϕ is self-financing, then the discounted value process satisfies

$$D(0,t)V_t(\phi) = V_0(\phi) + \int_0^t \phi_u \, \mathrm{d}\big(D(0,u)S_u\big).$$

By Proposition 2.1.1 in the book, a strategy ϕ is self-financing if and only if

$$D(0,t)V_t(\phi) = V_0(\phi) + \int_0^t \phi_u \, \mathrm{d}\big(D(0,u)S_u\big).$$

Intuitively, dividing by the bank account rewrites all asset values in discounted units. Since self-financing means "no external cash flows," the only source of variation in the discounted portfolio value is the cumulative trading gain from the discounted asset-price processes themselves.

(c) Explain why this discounted form is conceptually important for no-arbitrage pricing.

It is important because no-arbitrage theory is naturally expressed in discounted units. Under an equivalent martingale measure, the discounted traded asset prices are martingales. Once the portfolio value is written in discounted form, pricing and replication become statements about martingales and conditional expectations rather than about raw asset prices with deterministic growth from the money-market account.

Problem A-2. Equivalent martingale measures and attainable claims

Suppose $\mathbb{Q}$ is an equivalent martingale measure, so that the discounted asset-price process $D(0,\cdot)S$ is a martingale under $\mathbb{Q}$. Let H be an attainable contingent claim.

(a) State the pricing formula for the unique no-arbitrage price process π_t associated with H.

Proposition 2.1.2 states that for each $0 \le t \le T$, the unique price is

$$\pi_t = \mathbb{E}(D(t,T)H \mid \mathcal{F}_t). \tag{2.2}$$

(b) Take the particular claim $H \equiv 1$ paid at time T. Show that the pricing formula reduces to

$$\pi_t = P(t,T).$$

If $H \equiv 1$, then

$$\pi_t = \mathbb{E}(D(t,T) \mid \mathcal{F}_t).$$

But the claim paying one unit at time T is precisely a T-maturity zero-coupon bond. Hence its time-t price is, by definition, $P(t,T)$. Therefore

$$\pi_t = P(t,T).$$

So the general pricing formula correctly nests the basic bond-pricing case.

(c) Explain why attainability is the key hypothesis behind uniqueness of price here.

If a claim is attainable, there exists a self-financing strategy that replicates its terminal payoff exactly. Then any no-arbitrage price must equal the value of that replicating strategy; otherwise one could construct an arbitrage by buying the cheaper object and selling the more expensive one. Thus attainability is what ties the claim to a specific trading strategy and therefore forces uniqueness of price.

Problem A-3. Numeraires and the Radon–Nikodym derivative

Let N be a numeraire with associated measure $\mathbb{Q}^N$, and let U be another numeraire with associated measure $\mathbb{Q}^U$. Proposition 2.2.1 states that for any attainable claim Y,

$$\frac{Y_t}{U_t} = \mathbb{E}_t^U\left[\frac{Y_T}{U_T}\right],$$

and that the Radon–Nikodym derivative is

$$\frac{\mathrm{d}\mathbb{Q}^U}{\mathrm{d}\mathbb{Q}^N} = \frac{U_T N_0}{U_0 N_T}. \tag{2.5}$$

(a) Explain what a numeraire is, and why changing numeraires may simplify pricing.

A numeraire is any positive non-dividend-paying asset. Choosing a numeraire means measuring every asset price relative to that asset. This can simplify pricing because some payoffs become much more natural when normalized by a specially chosen asset. In interest-rate problems, the bank account is always valid, but a zero-coupon bond or swap annuity often matches the payoff structure better and removes inconvenient discounting terms from the expectation.

(b) Starting from the identity

$$\mathbb{E}^N\left[\frac{Z_T}{N_T}\right] = \mathbb{E}^U\left[\frac{U_0 Z_T}{N_0 U_T}\right], \tag{2.6}$$

explain how formula (2.5) arises.

By definition of the Radon–Nikodym derivative, for any integrable random variable X one has

$$\mathbb{E}^N[X] = \mathbb{E}^U\left[X\,\frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{Q}^U}\right].$$

Apply this with $X = Z_T/N_T$. Then

$$\mathbb{E}^N\left[\frac{Z_T}{N_T}\right] = \mathbb{E}^U\left[\frac{Z_T}{N_T}\frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{Q}^U}\right].$$

Comparing this with (2.6), and using the arbitrariness of Z_T, yields

$$\frac{1}{N_T}\frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{Q}^U} = \frac{U_0}{N_0 U_T}.$$

Rearranging gives

$$\frac{\mathrm{d}\mathbb{Q}^U}{\mathrm{d}\mathbb{Q}^N} = \frac{U_T N_0}{U_0 N_T},$$

which is exactly formula (2.5).

(c) Explain, in words, why changing the numeraire changes the measure but not the price.

The claim being priced is the same economic object regardless of the coordinate system used to describe it. Changing the numeraire only changes the unit in which values are expressed and therefore changes the probability measure under which normalized prices are martingales. The price itself cannot depend on that representational choice, because otherwise the same claim would have two different market values.

Problem B-1. Fact One and the natural martingale examples

(a) Explain why an asset price divided by its chosen numeraire should have zero drift under the measure associated with that numeraire.

By Proposition 2.2.1, once a numeraire is chosen, any attainable claim normalized by that numeraire is a martingale under the associated measure. A martingale has no drift in its semimartingale decomposition. Therefore the ratio “asset / numeraire” must have zero drift under that numeraire-associated measure.

(b) Use this principle to explain why the forward LIBOR rate

$$F_2(t) = \frac{(P(t,T_1) - P(t,T_2))/(T_2 - T_1)}{P(t,T_2)}$$

is a martingale under the T_2-forward measure.

The numerator is a tradable portfolio of zero-coupon bonds, and the denominator $P(t, T_2)$ is itself a strictly positive tradable asset, hence a valid numeraire. Therefore $F_2(t)$ is a traded-asset portfolio divided by the numeraire $P(t, T_2)$. By Fact One, such a ratio must be a martingale under the measure associated with $P(\cdot, T_2)$, namely the T_2-forward measure.

(c) Use the same principle to explain why the forward swap rate

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta}(T_i - T_{i-1})P(t, T_i)}$$

is a martingale under the swap measure associated with the annuity

$$\sum_{i=\alpha+1}^{\beta}(T_i - T_{i-1})P(t, T_i).$$

Here the numerator is again a tradable portfolio, long one T_α bond and short one T_β bond. The denominator is the positive bond portfolio defining the swap annuity. Since the forward swap rate is exactly "tradable asset divided by chosen numeraire," Fact One implies that it is a martingale under the measure associated with that annuity, namely the swap measure.

Problem B-2. Fact Two and numeraire invariance of price

Fact Two states that

$$\text{Price}_t = \mathbb{E}_t^B\left[\frac{B(t)}{B(T)}\text{Payoff}(T)\right] = \mathbb{E}_t^S\left[\frac{S_t}{S_T}\text{Payoff}(T)\right].$$

(a) Explain why this identity is the right formal expression of "same claim, same price."

The left-hand and right-hand expressions price the same terminal payoff, but under two different numeraires and therefore under two different associated measures. Equality means that changing coordinates changes the representation of the expectation, not the value of the claim. This is precisely the formal statement of "same claim, same price."

(b) In the special case $\text{Payoff}(T) = 1$, explain how the formula reproduces the price of a zero-coupon bond under any convenient numeraire.

If the payoff is 1 at time T, the claim is a zero-coupon bond. Hence

$$P(t, T) = \mathbb{E}_t^B\left[\frac{B(t)}{B(T)}\right] = \mathbb{E}_t^S\left[\frac{S_t}{S_T}\right].$$

So the bond price can be computed under the bank-account measure or under any other convenient numeraire-associated measure, provided the compensating numeraire ratio is used inside the expectation.

(c) Explain why the bank-account numeraire is correct but often inconvenient in interest-rate problems.

It is correct because the bank account is always a valid positive numeraire, and the corresponding measure is the familiar risk-neutral measure. It is often inconvenient because the discount factor $B(t)/B(T)$ can remain stochastic and strongly intertwined with the payoff. In interest-rate derivatives, a bond numeraire or annuity numeraire often aligns much more naturally with the payoff and produces simpler martingale dynamics for the relevant rate.

Problem B-3. From Girsanov to the drift change formula

Let X be an n-vector diffusion. Under the measure associated with numeraire S, suppose

$$\mathrm{d}X_t = \mu_t^S(X_t)\,\mathrm{d}t + \sigma_t(X_t)C\,\mathrm{d}W_t^S,$$

and under the measure associated with numeraire U suppose

$$\mathrm{d}X_t = \mu_t^U(X_t)\,\mathrm{d}t + \sigma_t(X_t)C\,\mathrm{d}W_t^U.$$

Assume further that under $\mathbb{Q}^U$,

$$\mathrm{d}S_t = (\cdots)\mathrm{d}t + \sigma_t^S C\,\mathrm{d}W_t^U, \qquad \mathrm{d}U_t = (\cdots)\mathrm{d}t + \sigma_t^U C\,\mathrm{d}W_t^U.$$

(a) Starting from

$$\zeta_t = \frac{U_0 S_t}{S_0 U_t}, \tag{2.9}$$

explain why ζ is the density process relating $\mathbb{Q}^S$ and $\mathbb{Q}^U$.

By formula (2.5), the Radon–Nikodym derivative between the two numeraire-associated measures is obtained by the ratio of terminal numeraire values multiplied by the initial values. At time t, the corresponding density process is therefore

$$\zeta_t = \frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}^U}\Big|_{\mathcal{F}_t} = \frac{U_0 S_t}{S_0 U_t}.$$

Thus ζ is exactly the process that converts $\mathbb{Q}^U$-expectations into $\mathbb{Q}^S$-expectations.

(b) Use the martingale dynamics of S_t/U_t under $\mathbb{Q}^U$ to derive

$$\mathrm{d}\zeta_t = \frac{U_0}{S_0}\,\frac{S_t}{U_t}\,\sigma_t^{S/U} C\,\mathrm{d}W_t^U. \tag{2.10}$$

Under $\mathbb{Q}^U$, the ratio S_t/U_t is a martingale by Fact One, so it has dynamics

$$\mathrm{d}\left(\frac{S_t}{U_t}\right) = \sigma_t^{S/U} C\,\mathrm{d}W_t^U.$$

Since $\zeta_t = (U_0/S_0)(S_t/U_t)$, multiplying the last display by U_0/S_0 yields

$$\mathrm{d}\zeta_t = \frac{U_0}{S_0}\,\frac{S_t}{U_t}\,\sigma_t^{S/U} C\,\mathrm{d}W_t^U,$$

which is exactly (2.10).

(c) By comparing this with the exponential-martingale dynamics

$$\mathrm{d}\zeta_t = \alpha_t \zeta_t \,\mathrm{d}W_t^U, \tag{2.7}$$

derive

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \sigma_t(X_t)\,\rho\left(\frac{\sigma_t^S}{S_t} - \frac{\sigma_t^U}{U_t}\right)', \tag{2.12}$$

where $\rho = CC'$. You may quote formula (2.11) as an intermediate step.

Comparing (2.7) and (2.10) shows that

$$\alpha_t \zeta_t = \frac{U_0}{S_0}\frac{S_t}{U_t}\,\sigma_t^{S/U} C.$$

Using $\zeta_t = (U_0 S_t)/(S_0 U_t)$, this simplifies to

$$\alpha_t = \sigma_t^{S/U} C.$$

By definition of α_t, one has

$$\alpha_t = (\mu_t^S(X_t) - \mu_t^U(X_t))(\sigma_t(X_t)C)^{-1}.$$

Substituting the expression for α_t and rearranging gives formula (2.11):

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \frac{U_t}{S_t}\,\sigma_t(X_t)\rho(\sigma_t^{S/U})'. \tag{2.11}$$

Now apply the stochastic Leibniz rule to S_t/U_t. Under $\mathbb{Q}^U$, the diffusion coefficient of the ratio is

$$\sigma_t^{S/U} = \frac{1}{U_t}\sigma_t^S - \frac{S_t}{U_t^2}\sigma_t^U = \frac{S_t}{U_t}\left(\frac{\sigma_t^S}{S_t} - \frac{\sigma_t^U}{U_t}\right).$$

Insert this into (2.11). The factor (U_t/S_t) cancels with (S_t/U_t), leaving

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \sigma_t(X_t)\rho\left(\frac{\sigma_t^S}{S_t} - \frac{\sigma_t^U}{U_t}\right)',$$

which is exactly formula (2.12).

Problem B-4. Brownian recentering and economic interpretation

This problem is conceptual rather than computational.

(a) Explain why a Brownian motion that is driftless under one numeraire-associated measure need not remain driftless after the numeraire is changed.

A Brownian motion is defined relative to a particular probability measure. When the numeraire changes, the associated probability measure changes as well. Under the new measure, the old shock process generally acquires a predictable drift term, so it is no longer centered in the sense required of a Brownian motion.

(b) Explain, in words, what is meant by "re-centering the Brownian noise."

Re-centering means subtracting off the drift that the old shock picks up under the new measure so that the resulting process once again has mean-zero local increments. After this correction, the new process becomes the true Brownian motion for the new measure.

(c) Explain why this re-centering shows up in practice as a correction to the drift term of the asset dynamics.

The diffusion coefficient is an instantaneous object and does not change with the numeraire. What changes is the decomposition into drift plus centered noise. Once the old noise is rewritten as "new Brownian motion + correction," that correction moves into the $\mathrm{d}t$ term. Therefore the asset keeps the same volatility loading but acquires a new drift.

(d) In a few precise sentences, summarize the economic meaning of formula (2.12).

Formula (2.12) says that changing the numeraire changes the drift of an asset by an amount determined by three ingredients: the asset's volatility exposure, the instantaneous correlation structure, and the difference between the volatility-normalized dynamics of the two numeraires. Economically, it is the precise correction needed so that prices expressed in units of the new numeraire become martingales under the new measure. In that sense, the formula is the bridge between "same price" and "different dynamics under different numeraires."