Interest Rate Models — Lecture 4 Notes

Sections 2.3.1–2.6.1: diffusion coefficients, convenient numeraires, forward measures, and pricing formulas

1 Completing the numeraire-change toolkit

Last lecture derived the drift correction that appears when the pricing measure is changed. If two numeraires S and U evolve under $\mathbb{Q}^U$ as

$$dS_t = (\cdots)dt + \sigma_t^S C\, dW_t^U, \qquad dU_t = (\cdots)dt + \sigma_t^U C\, dW_t^U, \qquad CC' = \rho,$$

then the drift of a diffusion X under the numeraire U is

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \sigma_t(X_t)\rho \left(\frac{\sigma_t^S}{S_t} - \frac{\sigma_t^U}{U_t}\right)'. \tag{2.12}$$

Thus changing measure does not change instantaneous volatility or instantaneous correlation; it changes the drift. Equivalently, the correlated Brownian shock is re-centered:

$$C\, dW_t^S = C\, dW_t^U - \rho \left(\frac{\sigma_t^S}{S_t} - \frac{\sigma_t^U}{U_t}\right)' dt. \tag{2.13}$$

The same adjustment can be read in two ways: a drift shift in the asset dynamics, or a re-centering of the Brownian noise under the new measure.

Level-proportional, lognormal-type dynamics

Assume volatilities are proportional to the level of the variables:

$$\sigma_t^S = v_t^S S_t, \qquad \sigma_t^U = v_t^U U_t, \qquad \sigma_t(X_t) = \operatorname{diag}(X_t)\operatorname{diag}(v_t^X).$$

Then (2.12) becomes

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \operatorname{diag}(X_t)\operatorname{diag}(v_t^X)\rho(v_t^S - v_t^U)'. \tag{2.14}$$

Equivalently,

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \operatorname{diag}(X_t)\frac{d\langle \ln X, \ln(S/U)\rangle_t}{dt}.$$

If X has deterministic proportional drift under $\mathbb{Q}^S$, so that

$$\mu_t^S(X_t) = \operatorname{diag}(X_t) m_t^S,$$

then it keeps the same level-proportional form under $\mathbb{Q}^U$:

$$\mu_t^U(X_t) = \operatorname{diag}(X_t) m_t^U, \qquad m_t^U dt = m_t^S dt - (d\ln X_t)(d\ln(S_t/U_t)). \tag{2.15}$$

So, in lognormal-type models, the numeraire change appears as a correction to the drift rate of $\ln X$.

2 The diffusion coefficient notation

For a scalar diffusion driven by the common correlated shock $C\, dW_t$,

$$dX_t = (\cdots)dt + v_t C\, dW_t,$$

define

$$\mathrm{DC}_t(X) = v_t.$$

The operator DC extracts the diffusion coefficient of the process. It is not merely a label for volatility; it is a manipulation device for the noise term.

With this notation, the drift-change and shock-change formulas become

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \mathrm{DC}_t(X)\rho\left(\frac{\mathrm{DC}_t(S)}{S_t} - \frac{\mathrm{DC}_t(U)}{U_t}\right)', \tag{2.16}$$

$$C\, dW_t^S = C\, dW_t^U - \rho\left(\frac{\mathrm{DC}_t(S)}{S_t} - \frac{\mathrm{DC}_t(U)}{U_t}\right)' dt. \tag{2.17}$$

The useful algebra is:

$$\mathrm{DC}(kX + hY) = k\,\mathrm{DC}(X) + h\,\mathrm{DC}(Y), \qquad \mathrm{DC}_t(g) = 0,$$

for deterministic k, h, g, and

$$\mathrm{DC}_t(g_1 X + g_2) = g_1(t)\,\mathrm{DC}_t(X).$$

For a scalar diffusion X,

$$\mathrm{DC}(\ln X) = \frac{\mathrm{DC}(X)}{X}, \qquad \mathrm{DC}(X) = X\,\mathrm{DC}(\ln X).$$

Therefore (2.17) can be written compactly as

$$C\,dW_t^S = C\,dW_t^U - ho\left(\mathrm{DC}\left(\ln\frac{S}{U}\right)\right)'\,dt. \tag{2.18}$$

The product rule follows from the logarithmic rule:

$$\mathrm{DC}(X_1\cdots X_m) = \left(\frac{\mathrm{DC}(X_1)}{X_1} + \cdots + \frac{\mathrm{DC}(X_m)}{X_m}\right) X_1\cdots X_m.$$

Finally, the quadratic covariation term

$$\frac{d\langle \ln X, \ln Y\rangle_t}{dt}$$

measures the instantaneous co-movement of the proportional shocks of X and Y.

3 Choosing a convenient numeraire

Suppose the claim pays $h(X_T)$ at time T. Under the money-market account numeraire, its time-zero value is

$$\mathbb{E}^0\{D(0,T)h(X_T)\}.$$

If S is another numeraire, the change-of-numeraire identity gives

$$\mathbb{E}^0\{D(0,T)h(X_T)\} = S_0\mathbb{E}^S\left\{\frac{h(X_T)}{S_T}\right\}. \tag{2.19}$$

This motivates the search for a numeraire S with two properties:

1. $X_t S_t$ is the price of a tradable asset;
2. $h(X_T)/S_T$ is simple enough to take expectations.

The first condition ensures that

$$\frac{X_t S_t}{S_t} = X_t$$

is a martingale under $\mathbb{Q}^S$. Thus one may model X as a driftless lognormal martingale under the measure attached to S:

$$dX_t = \sigma(t) X_t\,dW_t, \qquad \mathbb{Q}^S.$$

Then the distribution is known:

$$\ln X_t \sim N\left(\ln X_0 - \frac{1}{2}\int_0^t \sigma(s)^2 ds, \int_0^t \sigma(s)^2 ds\right).$$

The point is not to change the derivative price; it is to choose the unit of account that makes the payoff expectation tractable.

4 The forward measure

A central choice is the zero-coupon bond maturing at the derivative payoff date T. Since $P(T,T) = 1$, the payoff is not divided by a random terminal numeraire. The measure associated with $P(\cdot, T)$ is the T-forward measure, denoted $\mathbb{Q}^T$. If a claim pays H_T at time T, then

$$\pi_t = P(t,T)\mathbb{E}^T\{H_T \mid \mathcal{F}_t\}, \qquad 0 \le t \le T. \tag{2.20}$$

This is the fundamental practical formula: price equals the time-t bond price times a forward-measure expectation of the payoff.

Forward rates under the forward measure

For $0 \le u \le t \le S < T$, the simply compounded forward rate over $[S,T]$ satisfies

$$\mathbb{E}^T\{F(t;S,T) \mid \mathcal{F}_u\} = F(u;S,T).$$

In particular,

$$\mathbb{E}^T\{L(S,T) \mid \mathcal{F}_t\} = F(t;S,T). \tag{2.21}$$

The reason is structural:

$$F(t;S,T)P(t,T) = \frac{1}{\tau(S,T)}\big(P(t,S) - P(t,T)\big),$$

which is the price of a traded portfolio of zero-coupon bonds. Dividing by the numeraire $P(t,T)$ makes $F(t;S,T)$ a $\mathbb{Q}^T$-martingale.

The same idea holds in the instantaneous limit:

$$\mathbb{E}^T\{r_T \mid \mathcal{F}_t\} = f(t,T).$$

Thus the instantaneous forward rate is the T-forward-measure expectation of the future short rate at T.

5 Fundamental pricing formulas

For the rest of the book, attainable claims are priced under a risk-neutral measure by

$$\pi_t = \mathbb{E}\left[e^{-\int_t^T r_s ds} H_T \mid \mathcal{F}_t\right]. \tag{2.22}$$

For a European call option with maturity T, strike X, written on a unit-principal zero-coupon bond maturing at $S > T$,

$$ZBC(t,T,S,X) = \mathbb{E}\left[e^{-\int_t^T r_s ds}(P(T,S) - X)^+ \mid \mathcal{F}_t\right]. \tag{2.23}$$

The T-forward measure is defined from the risk-neutral measure by

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{P(T,T)B(0)}{P(0,T)B(T)} = \frac{D(0,T)}{P(0,T)}. \tag{2.24}$$

Since $P(T,T) = 1$, the random discounting is absorbed into the measure change. Therefore,

$$ZBC(t,T,S,X) = P(t,T)\mathbb{E}^T\left[(P(T,S) - X)^+ \mid \mathcal{F}_t\right]. \tag{2.25}$$

This form is useful whenever $P(T,S)$ has a known distribution under $\mathbb{Q}^T$, for example a lognormal distribution leading to a Black-like formula. The next step is to use these pricing ideas to rewrite caplets and floorlets as options on zero-coupon bonds.

Summary

- Changing numeraire changes the drift, or equivalently re-centers the Brownian shock.
- The notation $\mathrm{DC}(X)$ extracts the diffusion coefficient and lets us manipulate noise terms cleanly.
- A good numeraire makes the natural underlying a martingale and simplifies the payoff ratio.
- The T-forward measure uses $P(\cdot,T)$ as numeraire, giving $\pi_t = P(t,T)\mathbb{E}^T(H_T \mid \mathcal{F}_t)$.
- Forward rates are martingales under the forward measure attached to their payment date.
- Zero-coupon bond options can be rewritten under the forward measure, setting up cap/floor pricing.

Next lecture: pricing caps and floors by decomposing caplets and floorlets into zero-coupon bond options.