Interest Rate Models: Theory and Practice

Problem Set 4

Part A

Part A concerns diffusion coefficients, the DC operator, and the algebra needed for numeraire changes.

Problem A-1. DC extraction from concrete diffusions

Let $W = (W^1, W^2)$ be a two-dimensional Brownian motion with instantaneous correlation matrix

$$\rho = \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{pmatrix}.$$

Suppose the positive processes X and Y satisfy

$$dX_t = (0.04X_t + 0.02Y_t)dt + 0.18X_t\, dW_t^1 - 0.05Y_t\, dW_t^2,$$
$$dY_t = (0.03Y_t - 0.01X_t)dt + 0.07X_t\, dW_t^1 + 0.12Y_t\, dW_t^2.$$

In this problem, write diffusion coefficients as row vectors multiplying $(dW_t^1, dW_t^2)'$.

(a) Compute $\mathrm{DC}(X)$ and $\mathrm{DC}(Y)$.

(b) Compute $\mathrm{DC}(2X - 3Y)$.

(c) Explain why the drift terms do not enter the DC calculation.

Problem A-2. Logs, ratios, and instantaneous covariance

Continue with the processes X and Y from Problem A-1.

(a) Compute $\mathrm{DC}(\ln X)$ and $\mathrm{DC}(\ln Y)$.

(b) Compute

$$\mathrm{DC}\left(\ln \frac{X}{Y}\right).$$

(c) Using the correlation matrix ρ, express the instantaneous covariance term

$$d \ln X_t\, d \ln Y_t.$$

(d) Interpret this covariance term as the infinitesimal co-movement of the percentage changes of X and Y.

Problem A-3. What DC is actually doing

A student says: "Since $\mathrm{DC}(X)$ just gives the volatility term, DC is only another name for volatility."

(a) Explain why this statement is incomplete.

(b) Explain why it is better to think of DC as an operator that extracts and manipulates diffusion coefficients.

(c) Give one reason why this operator notation becomes useful in change-of-numeraire calculations.

Part B

Part B concerns numeraire choice, forward measures, and the first pricing applications.

Problem B-1. Choosing a convenient numeraire

For each object below, identify a natural numeraire and associated measure. Give a one- or two-sentence justification in each case.

(a) A claim paying H_T at time T.

(b) A caplet payoff paid at time S:

$$N\tau(T,S)(L(T,S)-K)^+.$$

(c) A swaption payoff naturally written in terms of the forward swap rate $S_{\alpha,\beta}$ and the swap annuity

$$C_{\alpha,\beta}(t)=\sum_{i=\alpha+1}^{\beta}\tau_i P(t,T_i).$$

(d) A payoff involving a ratio X_T/Y_T paid at time T.

Problem B-2. Correcting a forward-measure pricing mistake

Let H_T be an attainable payoff paid at time T. A classmate says:

$$\pi_t=\mathbb{E}_t^T[H_T].$$

(a) Explain what is missing from this formula.

(b) Starting from the risk-neutral formula

$$\pi_t=B(t)\mathbb{E}_t^Q\left[\frac{H_T}{B(T)}\right],$$

show that the correct T-forward-measure formula is

$$\pi_t=P(t,T)\mathbb{E}_t^T[H_T].$$

(c) Explain, in words, why changing the measure does not remove the numeraire from the pricing formula.

Problem B-3. Forward LIBOR under the S-forward measure

Let $t\le T<S$, and define the simply compounded forward rate

$$F(t;T,S)=\frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)}-1\right).$$

(a) Rewrite $F(t;T,S)$ as a traded bond portfolio divided by the numeraire $P(t,S)$.

(b) Explain why $F(t;T,S)$ is a martingale under the S-forward measure.

(c) Why is $P(t,S)$ more natural than $P(t,T)$ for a caplet whose payoff is paid at time S?

Problem B-4. Caplet pricing under the forward measure

Consider a caplet that resets at time T and pays at time S the amount

$$N\tau(T,S)(L(T,S)-K)^+.$$

(a) Use $L(T,S) = F(T;T,S)$ to rewrite the payoff in terms of the forward LIBOR rate.

(b) Derive the time-t pricing formula

$$\mathrm{Cpl}(t,T,S,K) = NP(t,S)\tau(T,S)\mathbb{E}_t^S\left[(F(T;T,S)-K)^+\right].$$

(c) Explain why the S-forward measure is the natural measure for this caplet.

(d) Explain why pricing the same caplet under the bank-account numeraire is correct but less clean.