Interest Rate Models: Theory and Practice

Solutions to Problem Set 4

Problem A-1. DC extraction from concrete diffusions

Let $W = (W^1, W^2)$ be a two-dimensional Brownian motion with instantaneous correlation matrix

$$\rho = \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{pmatrix}.$$

Suppose the positive processes X and Y satisfy

$$dX_t = (0.04X_t + 0.02Y_t)dt + 0.18X_t\, dW_t^1 - 0.05Y_t\, dW_t^2,$$

$$dY_t = (0.03Y_t - 0.01X_t)dt + 0.07X_t\, dW_t^1 + 0.12Y_t\, dW_t^2.$$

In this problem, write diffusion coefficients as row vectors multiplying $(dW_t^1, dW_t^2)'$.

(a) Compute $\mathrm{DC}(X)$ and $\mathrm{DC}(Y)$.

(b) Compute $\mathrm{DC}(2X - 3Y)$.

(c) Explain why the drift terms do not enter the DC calculation.

Solution.

(a) The diffusion coefficient is the row vector multiplying the Brownian increment. Therefore

$$\mathrm{DC}(X) = (0.18X_t, -0.05Y_t),$$

while

$$\mathrm{DC}(Y) = (0.07X_t, 0.12Y_t).$$

(b) By linearity of the DC operator,

$$\mathrm{DC}(2X - 3Y) = 2\,\mathrm{DC}(X) - 3\,\mathrm{DC}(Y).$$

Hence

$$\mathrm{DC}(2X - 3Y) = 2(0.18X_t, -0.05Y_t) - 3(0.07X_t, 0.12Y_t),$$

so

$$\mathrm{DC}(2X - 3Y) = (0.15X_t, -0.46Y_t).$$

(c) The DC operator extracts only the diffusion coefficient, meaning the coefficient of the Brownian noise. The drift terms multiply dt, not dW_t, so they are irrelevant for the DC calculation. This is exactly why DC is useful: it isolates the instantaneous noise exposure of the process.

Problem A-2. Logs, ratios, and instantaneous covariance

Continue with the processes X and Y from Problem A-1.

(a) Compute $\mathrm{DC}(\ln X)$ and $\mathrm{DC}(\ln Y)$.

(b) Compute

$$\mathrm{DC}\left(\ln \frac{X}{Y}\right).$$

(c) Using the correlation matrix ρ, express the instantaneous covariance term

$$d \ln X_t \, d \ln Y_t.$$

(d) Interpret this covariance term as the infinitesimal co-movement of the percentage changes of X and Y.

Solution.

(a) For a positive process X,

$$\mathrm{DC}(\ln X) = \frac{\mathrm{DC}(X)}{X}.$$

Thus

$$\mathrm{DC}(\ln X) = \left(0.18, -0.05\frac{Y_t}{X_t}\right).$$

Similarly,

$$\mathrm{DC}(\ln Y) = \frac{\mathrm{DC}(Y)}{Y} = \left(0.07\frac{X_t}{Y_t}, 0.12\right).$$

(b) Since

$$\ln \frac{X}{Y} = \ln X - \ln Y,$$

linearity gives

$$\mathrm{DC}\left(\ln \frac{X}{Y}\right) = \mathrm{DC}(\ln X) - \mathrm{DC}(\ln Y).$$

Therefore

$$\mathrm{DC}\left(\ln \frac{X}{Y}\right) = \left(0.18 - 0.07\frac{X_t}{Y_t}, -0.05\frac{Y_t}{X_t} - 0.12\right).$$

(c) Let

$$a_t = \mathrm{DC}(\ln X) = \left(0.18, -0.05\frac{Y_t}{X_t}\right), \qquad b_t = \mathrm{DC}(\ln Y) = \left(0.07\frac{X_t}{Y_t}, 0.12\right).$$

Then

$$d \ln X_t \, d \ln Y_t = a_t \rho b_t' \, dt.$$

Writing this out,

$$d \ln X_t \, d \ln Y_t = \left[0.18\left(0.07\frac{X_t}{Y_t}\right) + 0.18\rho_{12}(0.12) - 0.05\frac{Y_t}{X_t}\rho_{12}\left(0.07\frac{X_t}{Y_t}\right) - 0.05\frac{Y_t}{X_t}(0.12)\right] dt.$$

Equivalently,

$$d \ln X_t \, d \ln Y_t = \left[0.0126\frac{X_t}{Y_t} + 0.0216\rho_{12} - 0.0035\rho_{12} - 0.006\frac{Y_t}{X_t}\right] dt.$$

(d) Since $d \ln X_t$ and $d \ln Y_t$ represent infinitesimal percentage changes, the term $d \ln X_t \, d \ln Y_t$ measures their instantaneous co-movement. The expression depends not only on the individual diffusion loadings of X and Y, but also on the instantaneous correlation structure of the Brownian shocks.

Problem A-3. What DC is actually doing

A student says: "Since $\mathrm{DC}(X)$ just gives the volatility term, DC is only another name for volatility."

(a) Explain why this statement is incomplete.

(b) Explain why it is better to think of DC as an operator that extracts and manipulates diffusion coefficients.

(c) Give one reason why this operator notation becomes useful in change-of-numeraire calculations.

Solution.

(a) The statement is incomplete because DC is not merely a label for volatility. It extracts the full Brownian loading of a process, including how the process is exposed to each Brownian driver. In a multi-dimensional model this is usually a vector, not a single volatility number.

(b) It is better to think of DC as an operator because it obeys useful algebraic rules. For example, it is linear:

$$\mathrm{DC}(aX + bY) = a\,\mathrm{DC}(X) + b\,\mathrm{DC}(Y),$$

and for positive X,

$$\mathrm{DC}(\ln X) = \frac{\mathrm{DC}(X)}{X}.$$

Thus DC lets us manipulate diffusion coefficients without rewriting the entire stochastic differential equation each time.

(c) Change-of-numeraire formulas involve quantities such as the diffusion coefficient of ratios and logarithms of numeraires. DC notation makes these calculations compact. Instead of repeatedly applying Ito's formula from scratch, one can use DC rules to identify the relevant Brownian exposures directly.

Problem B-1. Choosing a convenient numeraire

For each object below, identify a natural numeraire and associated measure. Give a one- or two-sentence justification in each case.

(a) A claim paying H_T at time T.

(b) A caplet payoff paid at time S:

$$N\tau(T,S)(L(T,S) - K)^+.$$

(c) A swaption payoff naturally written in terms of the forward swap rate $S_{\alpha,\beta}$ and the swap annuity

$$C_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i).$$

(d) A payoff involving a ratio X_T/Y_T paid at time T.

Solution.

(a) A natural numeraire is the T-maturity zero-coupon bond $P(t,T)$, with associated T-forward measure Q^T. This is natural because the payoff is paid exactly at time T, so the pricing formula becomes

$$\pi_t = P(t,T)\mathbb{E}_t^T[H_T].$$

(b) A natural numeraire is $P(t,S)$, with associated S-forward measure. The caplet pays at time S, and the underlying forward LIBOR rate $F(t;T,S)$ is naturally written in terms of the bond ratio $P(t,T)/P(t,S)$.

(c) A natural numeraire is the swap annuity

$$C_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i),$$

with associated swap measure. Under this measure, the forward swap rate is the natural martingale, which is exactly the rate entering the swaption payoff.

(d) If the payoff involves a ratio X_T/Y_T, the natural numeraire may be Y if Y is a positive traded asset. Measuring X in units of Y makes the ratio X/Y the normalized object, and under the Y-associated measure this normalized traded price is a martingale.

Problem B-2. Correcting a forward-measure pricing mistake

Let H_T be an attainable payoff paid at time T. A classmate says:

$$\pi_t = \mathbb{E}_t^T[H_T].$$

(a) Explain what is missing from this formula.

(b) Starting from the risk-neutral formula

$$\pi_t = B(t)\mathbb{E}_t^Q\left[\frac{H_T}{B(T)}\right],$$

show that the correct T-forward-measure formula is

$$\pi_t = P(t,T)\mathbb{E}_t^T[H_T].$$

(c) Explain, in words, why changing the measure does not remove the numeraire from the pricing formula.

Solution.

(a) The formula is missing the time-t value of the numeraire, namely $P(t,T)$. Under the T-forward measure, the expectation $\mathbb{E}_t^T[H_T]$ is a forward value in units of the T-bond. To convert it into a currency price, one must multiply by $P(t,T)$.

(b) The general numeraire-pricing identity says that if N is the numeraire associated with Q^N, then

$$\frac{\pi_t}{N_t} = \mathbb{E}_t^N\left[\frac{\pi_T}{N_T}\right].$$

Choose $N_t = P(t,T)$. Since the claim pays H_T at time T and $P(T,T) = 1$, we have $N_T = 1$ and $\pi_T = H_T$. Therefore

$$\frac{\pi_t}{P(t,T)} = \mathbb{E}_t^T[H_T],$$

so

$$\pi_t = P(t,T)\mathbb{E}_t^T[H_T].$$

This is the same value as the risk-neutral expression

$$\pi_t = B(t)\mathbb{E}_t^Q\left[\frac{H_T}{B(T)}\right],$$

but represented under the bond numeraire rather than the bank-account numeraire.

(c) Changing the measure changes which normalized price is a martingale. It does not make the unit of account disappear. The expectation under Q^T gives the value measured in units of the T-bond, so multiplying by $P(t,T)$ converts that normalized value back into the actual time-t price.

Problem B-3. Forward LIBOR under the S-forward measure

Let $t \le T < S$, and define the simply compounded forward rate

$$F(t;T,S) = \frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right).$$

(a) Rewrite $F(t;T,S)$ as a traded bond portfolio divided by the numeraire $P(t,S)$.

(b) Explain why $F(t;T,S)$ is a martingale under the S-forward measure.

(c) Why is $P(t,S)$ more natural than $P(t,T)$ for a caplet whose payoff is paid at time S?

Solution.

(a) We can rewrite the forward rate as

$$F(t;T,S) = \frac{1}{\tau(T,S)}\frac{P(t,T) - P(t,S)}{P(t,S)}.$$

The numerator $P(t,T) - P(t,S)$ is the price of a traded bond portfolio: long one T-maturity bond and short one S-maturity bond. The denominator $P(t,S)$ is the chosen numeraire.

(b) Under the measure associated with $P(\cdot,S)$, any traded asset price divided by $P(\cdot,S)$ is a martingale. Since $F(t;T,S)$ is a constant multiple of a traded bond portfolio divided by $P(t,S)$, it follows that $F(t;T,S)$ is a martingale under the S-forward measure.

(c) The caplet cash flow is paid at time S, not time T. Also, the forward LIBOR rate for the period $[T,S]$ is naturally represented through the ratio $P(t,T)/P(t,S)$. Therefore $P(t,S)$ matches both the payment date and the bond-ratio structure of the underlying rate. The T-bond is connected to the reset date, but the payoff is settled at S.

Problem B-4. Caplet pricing under the forward measure

Consider a caplet that resets at time T and pays at time S the amount

$$N\tau(T,S)(L(T,S) - K)^+.$$

(a) Use $L(T,S) = F(T;T,S)$ to rewrite the payoff in terms of the forward LIBOR rate.

(b) Derive the time-t pricing formula

$$\mathrm{Cpl}(t,T,S,K) = NP(t,S)\tau(T,S)\mathbb{E}_t^S\left[(F(T;T,S) - K)^+\right].$$

(c) Explain why the S-forward measure is the natural measure for this caplet.

(d) Explain why pricing the same caplet under the bank-account numeraire is correct but less clean.

Solution.

(a) At the reset time T, the spot LIBOR rate for $[T, S]$ equals the forward rate observed at T for the same period:

$$L(T, S) = F(T; T, S).$$

Thus the caplet payoff at time S is

$$N\tau(T, S)(F(T; T, S) - K)^+.$$

(b) Since the payoff is paid at time S, choose the numeraire $P(t, S)$. The general forward-measure pricing formula gives

$$\text{Cpl}(t, T, S, K) = P(t, S)\mathbb{E}_t^S \left[N\tau(T, S)(F(T; T, S) - K)^+ \right].$$

Since N and $\tau(T, S)$ are deterministic contract terms, they can be pulled outside the expectation:

$$\text{Cpl}(t, T, S, K) = NP(t, S)\tau(T, S)\mathbb{E}_t^S \left[(F(T; T, S) - K)^+ \right].$$

(c) The S-forward measure is natural because the cash flow is paid at time S, and the underlying rate $F(t; T, S)$ is a martingale under this measure. Therefore the caplet becomes an option on a forward rate under the measure in which that forward rate has no drift.

(d) Pricing under the bank-account numeraire is still correct:

$$\text{Cpl}(t, T, S, K) = B(t)\mathbb{E}_t^Q \left[\frac{N\tau(T, S)(F(T; T, S) - K)^+}{B(S)} \right].$$

However, this expression keeps the stochastic discount factor inside the expectation. The S-forward measure replaces that stochastic discounting object with the outside factor $P(t, S)$, giving a cleaner expression and more convenient dynamics for the forward LIBOR rate.