Interest Rate Models — Lecture 5 Notes

Sections 2.6.1–2.9: caps/floors, deferred payoffs, multiple payoffs, and foreign numeraires

1 Pricing caps and floors through bond options

A European zero-coupon bond call with maturity T, strike X, and underlying bond maturity $S > T$ is

$$ZBC(t,T,S,X) = P(t,T)\mathbb{E}^T\left[(P(T,S)-X)^+ \mid \mathcal{F}_t\right]. \tag{2.25}$$

Let d_i be the cap/floor payment dates, t_i the corresponding times, τ_i the year fraction from d_{i-1} to d_i, and N the nominal. The i-th caplet pays at t_i but depends on the rate fixed at t_{i-1}:

$$\begin{aligned} Cpl(t,t_{i-1},t_i,\tau_i,N,X) &= \mathbb{E}\left[e^{-\int_t^{t_i} r_s ds} N\tau_i (L(t_{i-1},t_i)-X)^+ \mid \mathcal{F}_t\right] \\ &= N\mathbb{E}\left[e^{-\int_t^{t_{i-1}} r_s ds} P(t_{i-1},t_i)\tau_i (L(t_{i-1},t_i)-X)^+ \mid \mathcal{F}_t\right]. \end{aligned}$$

The second line anticipates the payoff to the reset date t_{i-1} by multiplying by $P(t_{i-1},t_i)$; this is the deferred-payoff idea of Section 2.7. Using

$$L(t_{i-1},t_i) = \frac{1-P(t_{i-1},t_i)}{\tau_i P(t_{i-1},t_i)},$$

one obtains

$$Cpl(t,t_{i-1},t_i,\tau_i,N,X) = N\mathbb{E}\left[e^{-\int_t^{t_{i-1}} r_s ds}\big(1-(1+X\tau_i)P(t_{i-1},t_i)\big)^+ \mid \mathcal{F}_t\right].$$

Define

$$X_i' = \frac{1}{1+X\tau_i}, \qquad N_i' = N(1+X\tau_i).$$

Then

$$Cpl(t,t_{i-1},t_i,\tau_i,N,X) = N_i' ZBP(t,t_{i-1},t_i,X_i'). \tag{2.26}$$

Equivalently, the same caplet can be written as

$$Cpl(t,t_{i-1},t_i,\tau_i,N,X) = ZBP(t,t_{i-1},t_i,N_i',N). \tag{2.27}$$

The analogous floorlet formulas are

$$Fll(t,t_{i-1},t_i,\tau_i,N,X) = N_i' ZBC(t,t_{i-1},t_i,X_i') = ZBC(t,t_{i-1},t_i,N_i',N). \tag{2.28}$$

Thus a caplet is a multiple of a bond put, and a floorlet is a multiple of a bond call.

Caps and floors are obtained by summing over the component caplets and floorlets:

$$\begin{aligned} Cap(t,T,\tau,N,X) &= \sum_{i=1}^{n} N_i' ZBP(t,t_{i-1},t_i,X_i'), \\ Flr(t,T,\tau,N,X) &= \sum_{i=1}^{n} N_i' ZBC(t,t_{i-1},t_i,X_i'). \end{aligned} \tag{2.29}$$

The point is structural: cap/floor pricing reduces to bond-option pricing once the reset/payment timing is handled correctly.

2 Pricing claims with deferred payoffs

Let $t < \tau < T$, and let H_τ be known at time τ but paid at time T. Then

$$\begin{aligned} \pi_t &= \mathbb{E}\left[e^{-\int_t^T r_s ds} H_\tau \mid \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\mathbb{E}\left(e^{-\int_t^T r_s ds} H_\tau \mid \mathcal{F}_\tau\right) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}\left[e^{-\int_t^\tau r_s ds} H_\tau \mathbb{E}\left(e^{-\int_\tau^T r_s ds} \mid \mathcal{F}_\tau\right) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}\left[e^{-\int_t^\tau r_s ds} H_\tau P(\tau,T) \mid \mathcal{F}_t\right]. \end{aligned}$$

By the tower property, the time-t value of H_τ paid at T equals the time-t value of the anticipated payoff

$$H_\tau P(\tau,T)$$

paid at τ. Economically, once the payoff is known at τ, paying at T is equivalent to paying its time-τ present value at τ.

3 Pricing claims with multiple payoffs

With several random cashflows at different dates, each cashflow could be priced under its own forward measure. For path-dependent products or Monte Carlo pricing, it is often better to work under one terminal measure. If H is $\mathcal{F}_T$-measurable and $t < T < S$, the forward-value counterpart of the deferred-payoff formula is

$$\mathbb{E}[D(t,T)H \mid \mathcal{F}_t] = \mathbb{E}\left[\frac{D(t,S)H}{P(T,S)} \mid \mathcal{F}_t\right]. \tag{2.30}$$

Thus a payoff H paid at T can be moved forward to date S by replacing it with

$$\frac{H}{P(T,S)}.$$

This is the forward value of the time-T payoff at the later time S.

For cashflows H_i paid at T_i, $T_1 < \cdots < T_n$, the direct price is

$$\pi_t = \sum_{i=1}^{n} \mathbb{E}\{D(t,T_i)H_i \mid \mathcal{F}_t\} = \sum_{i=1}^{n} P(t,T_i)\mathbb{E}^{T_i}\{H_i \mid \mathcal{F}_t\}.$$

Using (2.30), all cashflows can instead be moved to the terminal date T_n:

$$\pi_t = \sum_{i=1}^{n} \mathbb{E}\left\{\frac{D(t,T_n)H_i}{P(T_i,T_n)} \mid \mathcal{F}_t\right\} = P(t,T_n)\mathbb{E}^{T_n}\left\{\sum_{i=1}^{n} \frac{H_i}{P(T_i,T_n)} \mid \mathcal{F}_t\right\}.$$

This is why $\mathbb{Q}^{T_n}$ is often called the terminal forward measure.

4 Foreign markets and numeraire change

Section 2.9 is the first bridge to multi-currency interest-rate derivatives. Let X^f be a foreign-currency asset, B^f the foreign money-market account, B the domestic money-market account, and Q_t the exchange rate. There are two equivalent ways to value the same economic payoff:

- price X^f in the foreign market and convert the value into domestic currency at time t;
- convert the terminal payoff X_T^f into domestic currency at time T and price it domestically.

No arbitrage requires these two values to agree:

$$Q_t B_t^f \mathbb{E}^f\left(\frac{X_T^f}{B_T^f} \mid \mathcal{F}_t\right) = B_t \mathbb{E}\left(\frac{X_T^f Q_T}{B_T} \mid \mathcal{F}_t\right). \tag{2.31}$$

This identity is the domestic/foreign version of numeraire invariance: the object being priced is the same; only the currency and pricing measure have changed.

Summary

- Caplets and floorlets can be rewritten as zero-coupon bond options.
- Deferred payoffs separate into a payoff known at the reset date and a discounting adjustment to the payment date.
- Multiple payoff dates can be consolidated under one terminal forward measure by moving earlier cashflows forward.
- Foreign-market pricing is another numeraire-change problem: price abroad and convert, or convert the payoff and price domestically.