Interest Rate Models: Theory and Practice

Problem Set 5

Part A

Part A concerns caplet and floorlet decomposition, zero-coupon bond options, and the pricing of deferred payoffs.

Problem A-1. Caplets as zero-coupon bond puts

Let $t < t_{i-1} < t_i$. Consider a caplet with nominal N, accrual fraction τ_i, strike X, reset time t_{i-1}, and payment time t_i. Its time-t value is

$$\mathrm{Cpl}(t, t_{i-1}, t_i, \tau_i, N, X) = \mathbb{E}_t\left[e^{-\int_t^{t_i} r_s ds} N\tau_i \Big(L(t_{i-1}, t_i) - X\Big)^+\right].$$

(a) Explain the economic difference between the reset time t_{i-1} and the payment time t_i.

(b) Using iterated conditioning, rewrite the caplet value as

$$N\mathbb{E}_t\left[e^{-\int_t^{t_{i-1}} r_s ds} P(t_{i-1}, t_i)\tau_i \Big(L(t_{i-1}, t_i) - X\Big)^+\right].$$

(c) Use

$$L(t_{i-1}, t_i) = \frac{1 - P(t_{i-1}, t_i)}{\tau_i P(t_{i-1}, t_i)}$$

to show that

$$P(t_{i-1}, t_i)\tau_i \Big(L(t_{i-1}, t_i) - X\Big)^+ = \Big[1 - (1 + X\tau_i)P(t_{i-1}, t_i)\Big]^+.$$

(d) Define

$$X_i' = \frac{1}{1 + X\tau_i}, \qquad N_i' = N(1 + X\tau_i).$$

Show that the caplet can be written as a multiple of a zero-coupon bond put:

$$\mathrm{Cpl}(t, t_{i-1}, t_i, \tau_i, N, X) = N_i' \,\mathrm{ZBP}(t, t_{i-1}, t_i, X_i').$$

Problem A-2. Floorlets as zero-coupon bond calls

Consider the corresponding floorlet, with time-t value

$$\mathrm{Fll}(t, t_{i-1}, t_i, \tau_i, N, X) = \mathbb{E}_t\left[e^{-\int_t^{t_i} r_s ds} N\tau_i \Big(X - L(t_{i-1}, t_i)\Big)^+\right].$$

(a) Use iterated conditioning to move the payment from t_i back to t_{i-1}.

(b) Show that the floorlet payoff can be written in the form

$$N\Big[(1 + X\tau_i)P(t_{i-1}, t_i) - 1\Big]^+.$$

(c) Using the same definitions of X_i' and N_i' as in Problem A-1, show that

$$\mathrm{Fll}(t, t_{i-1}, t_i, \tau_i, N, X) = N_i' \, \mathrm{ZBC}(t, t_{i-1}, t_i, X_i').$$

(d) Explain why caps are portfolios of zero-coupon bond puts, while floors are portfolios of zero-coupon bond calls.

Problem A-3. Deferred payoffs

Let $t < \tau < T$, and suppose H_τ is known at time τ. Consider a claim that pays H_τ at the later date T.

(a) Starting from the risk-neutral pricing formula,

$$\pi_t = \mathbb{E}_t \left[e^{-\int_t^T r_s ds} H_\tau \right],$$

use the tower property of conditional expectation to show that

$$\pi_t = \mathbb{E}_t \left[e^{-\int_t^\tau r_s ds} H_\tau P(\tau, T) \right].$$

(b) Explain why this means that a payoff known at τ but paid at T may be treated as an anticipated payoff $H_\tau P(\tau, T)$ paid at τ.

(c) Explain how this idea helps interpret the caplet formulas from Problems A-1 and A-2.

Part B

Part B concerns multiple payoff dates, terminal forward measures, and the foreign-market numeraire bridge.

Problem B-1. Moving a payoff forward to a common terminal date

Let $t < T < S$, and let H be known at time T. Proposition 2.8.1 states that

$$\mathbb{E}_t[D(t, T)H] = \mathbb{E}_t \left[\frac{D(t, S)H}{P(T, S)} \right]. \tag{2.30}$$

(a) Prove this identity using $D(t, S) = D(t, T)D(T, S)$ and the tower property.

(b) Give the financial interpretation of the factor $1/P(T, S)$.

(c) Explain why this identity is useful for pricing products with many payoff dates.

Problem B-2. Multiple cash flows and the terminal measure

Let $T_1 < \cdots < T_n$, and suppose an interest-rate derivative pays H_i at time T_i, where H_i is known at T_i. Assume there are no early-exercise features and $t < T_1$.

(a) Write the time-t price as a sum of discounted expectations:

$$\pi_t = \sum_{i=1}^{n} \mathbb{E}_t[D(t, T_i)H_i].$$

(b) Rewrite the same price using the T_i-forward measure for each cash flow.

(c) Use Proposition 2.8.1 to rewrite all cash flows as if they were paid at the common terminal date T_n.

(d) Deduce the terminal-measure representation

$$\pi_t = P(t, T_n)\mathbb{E}_t^{T_n}\left[\sum_{i=1}^{n} \frac{H_i}{P(T_i, T_n)}\right].$$

(e) Explain why this form is especially convenient for Monte Carlo pricing.

Problem B-3. Foreign payoff, domestic value

Let X_T^f be a payoff denominated in foreign currency at time T. Let B^f be the foreign money-market account, B the domestic money-market account, and Q_t the exchange rate, defined as domestic currency per one unit of foreign currency.

(a) Write the time-t value of X_T^f in the foreign market under the foreign risk-neutral measure $\mathbb{Q}^f$.

(b) Convert this foreign value into domestic currency at time t.

(c) Now instead convert the future payoff into domestic currency at time T and price it under the domestic risk-neutral measure $\mathbb{Q}$.

(d) Explain why no-arbitrage requires

$$Q_t B_t^f \mathbb{E}_t^f\left[\frac{X_T^f}{B_T^f}\right] = B_t \mathbb{E}_t\left[\frac{X_T^f Q_T}{B_T}\right]. \tag{2.31}$$

(e) In words, summarize the difference between the two sides of (2.31).

Problem B-4. The foreign-to-domestic density

Using equation (2.31) at time 0, derive the candidate Radon–Nikodym derivative

$$\frac{d\mathbb{Q}^f}{d\mathbb{Q}} = \frac{Q_T B_T^f}{Q_0 B_T}. \tag{2.32}$$

Then explain why this measure change can be viewed as a numeraire change.