Interest Rate Models: Theory and Practice

Solutions to Problem Set 5

Problem A-1. Caplets as zero-coupon bond puts

Let $t < t_{i-1} < t_i$. Consider a caplet with nominal N, accrual fraction τ_i, strike X, reset time t_{i-1}, and payment time t_i. Its time-t value is

$$\mathrm{Cpl}(t, t_{i-1}, t_i, \tau_i, N, X) = \mathbb{E}_t\left[e^{-\int_t^{t_i} r_s ds} N\tau_i\Big(L(t_{i-1}, t_i) - X\Big)^+\right].$$

(a) Explain the economic difference between the reset time t_{i-1} and the payment time t_i.

The reset time t_{i-1} is the date on which the LIBOR rate $L(t_{i-1}, t_i)$ becomes known. The payment time t_i is the later date on which the caplet cash flow is actually exchanged. Thus the random amount is determined at t_{i-1}, but the cash is paid at t_i. This time lag is what makes the discounting step important.

(b) Using iterated conditioning, rewrite the caplet value as

$$N\mathbb{E}_t\left[e^{-\int_t^{t_{i-1}} r_s ds} P(t_{i-1}, t_i)\tau_i\Big(L(t_{i-1}, t_i) - X\Big)^+\right].$$

Start from

$$\mathrm{Cpl}(t, t_{i-1}, t_i, \tau_i, N, X) = N\mathbb{E}_t\left[e^{-\int_t^{t_i} r_s ds} \tau_i\Big(L(t_{i-1}, t_i) - X\Big)^+\right].$$

Use the decomposition

$$e^{-\int_t^{t_i} r_s ds} = e^{-\int_t^{t_{i-1}} r_s ds} e^{-\int_{t_{i-1}}^{t_i} r_s ds}.$$

Since $L(t_{i-1}, t_i)$ is known at t_{i-1}, the payoff factor

$$\tau_i\Big(L(t_{i-1}, t_i) - X\Big)^+$$

is $\mathcal{F}_{t_{i-1}}$-measurable. Therefore, by the tower property,

$$\begin{aligned}\mathrm{Cpl} &= N\mathbb{E}_t\left[e^{-\int_t^{t_{i-1}} r_s ds}\tau_i\Big(L(t_{i-1}, t_i) - X\Big)^+ \mathbb{E}_{t_{i-1}}\left(e^{-\int_{t_{i-1}}^{t_i} r_s ds}\right)\right] \\ &= N\mathbb{E}_t\left[e^{-\int_t^{t_{i-1}} r_s ds} P(t_{i-1}, t_i)\tau_i\Big(L(t_{i-1}, t_i) - X\Big)^+\right].\end{aligned}$$

(c) Use

$$L(t_{i-1}, t_i) = \frac{1 - P(t_{i-1}, t_i)}{\tau_i P(t_{i-1}, t_i)}$$

to show that

$$P(t_{i-1}, t_i)\tau_i\Big(L(t_{i-1}, t_i) - X\Big)^+ = \Big[1 - (1 + X\tau_i)P(t_{i-1}, t_i)\Big]^+.$$

Substitute the definition of the simply compounded LIBOR rate:

$$\tau_i(L(t_{i-1},t_i)-X)=\tau_i\left(\frac{1-P(t_{i-1},t_i)}{\tau_i P(t_{i-1},t_i)}-X\right).$$

This gives

$$\tau_i(L(t_{i-1},t_i)-X)=\frac{1-P(t_{i-1},t_i)}{P(t_{i-1},t_i)}-X\tau_i.$$

Multiplying by $P(t_{i-1},t_i)$ gives

$$P(t_{i-1},t_i)\tau_i(L(t_{i-1},t_i)-X)=1-P(t_{i-1},t_i)-X\tau_i P(t_{i-1},t_i),$$

so

$$P(t_{i-1},t_i)\tau_i(L(t_{i-1},t_i)-X)=1-(1+X\tau_i)P(t_{i-1},t_i).$$

Since $P(t_{i-1},t_i)>0$, the same relation holds inside the positive part:

$$P(t_{i-1},t_i)\tau_i\Big(L(t_{i-1},t_i)-X\Big)^+=\Big[1-(1+X\tau_i)P(t_{i-1},t_i)\Big]^+.$$

(d) Define

$$X_i'=\frac{1}{1+X\tau_i},\qquad N_i'=N(1+X\tau_i).$$

Show that the caplet can be written as a multiple of a zero-coupon bond put:

$$\mathrm{Cpl}(t,t_{i-1},t_i,\tau_i,N,X)=N_i'\,\mathrm{ZBP}(t,t_{i-1},t_i,X_i').$$

From part (c),

$$\mathrm{Cpl}=N\mathbb{E}_t\left[e^{-\int_t^{t_{i-1}}r_s ds}\Big[1-(1+X\tau_i)P(t_{i-1},t_i)\Big]^+\right].$$

Factor out $(1+X\tau_i)$ inside the positive part:

$$\Big[1-(1+X\tau_i)P(t_{i-1},t_i)\Big]^+=(1+X\tau_i)\left[\frac{1}{1+X\tau_i}-P(t_{i-1},t_i)\right]^+.$$

Using $X_i'=1/(1+X\tau_i)$ and $N_i'=N(1+X\tau_i)$,

$$\mathrm{Cpl}=N_i'\mathbb{E}_t\left[e^{-\int_t^{t_{i-1}}r_s ds}\Big(X_i'-P(t_{i-1},t_i)\Big)^+\right].$$

The expectation is exactly the price of a European put option with maturity t_{i-1}, strike X_i', and underlying zero-coupon bond $P(t_{i-1},t_i)$. Hence

$$\mathrm{Cpl}(t,t_{i-1},t_i,\tau_i,N,X)=N_i'\,\mathrm{ZBP}(t,t_{i-1},t_i,X_i').$$

Problem A-2. Floorlets as zero-coupon bond calls

Consider the corresponding floorlet, with time-t value

$$\mathrm{Fll}(t, t_{i-1}, t_i, \tau_i, N, X) = \mathbb{E}_t \left[e^{-\int_t^{t_i} r_s ds} N \tau_i \Big(X - L(t_{i-1}, t_i) \Big)^+ \right].$$

(a) Use iterated conditioning to move the payment from t_i back to t_{i-1}.

As in the caplet case, decompose the discount factor and condition first on $\mathcal{F}_{t_{i-1}}$. Since $L(t_{i-1}, t_i)$ is known at t_{i-1},

$$\mathrm{Fll} = N \mathbb{E}_t \left[e^{-\int_t^{t_{i-1}} r_s ds} P(t_{i-1}, t_i) \tau_i \Big(X - L(t_{i-1}, t_i) \Big)^+ \right].$$

Thus the future payment at t_i has been replaced by an anticipated payoff at t_{i-1} multiplied by the bond price $P(t_{i-1}, t_i)$.

(b) Show that the floorlet payoff can be written in the form

$$N \Big[(1 + X\tau_i) P(t_{i-1}, t_i) - 1 \Big]^+.$$

Using the LIBOR definition,

$$\tau_i (X - L(t_{i-1}, t_i)) = X\tau_i - \frac{1 - P(t_{i-1}, t_i)}{P(t_{i-1}, t_i)}.$$

Multiplying by $P(t_{i-1}, t_i)$ gives

$$P(t_{i-1}, t_i) \tau_i (X - L(t_{i-1}, t_i)) = X\tau_i P(t_{i-1}, t_i) - 1 + P(t_{i-1}, t_i).$$

Therefore

$$P(t_{i-1}, t_i) \tau_i (X - L(t_{i-1}, t_i)) = (1 + X\tau_i) P(t_{i-1}, t_i) - 1.$$

Taking positive parts and multiplying by N gives

$$N \Big[(1 + X\tau_i) P(t_{i-1}, t_i) - 1 \Big]^+.$$

(c) Using the same definitions of X_i' and N_i' as in Problem A-1, show that

$$\mathrm{Fll}(t, t_{i-1}, t_i, \tau_i, N, X) = N_i' \, \mathrm{ZBC}(t, t_{i-1}, t_i, X_i').$$

From part (b),

$$\mathrm{Fll} = N \mathbb{E}_t \left[e^{-\int_t^{t_{i-1}} r_s ds} \Big[(1 + X\tau_i) P(t_{i-1}, t_i) - 1 \Big]^+ \right].$$

Factor out $(1 + X\tau_i)$:

$$\Big[(1 + X\tau_i) P(t_{i-1}, t_i) - 1 \Big]^+ = (1 + X\tau_i) \left[P(t_{i-1}, t_i) - \frac{1}{1 + X\tau_i} \right]^+.$$

Thus

$$\text{Fll} = N_i'\mathbb{E}_t\left[e^{-\int_t^{t_{i-1}} r_s ds}\Big(P(t_{i-1},t_i) - X_i'\Big)^+\right].$$

The expectation is the price of a European call option on the zero-coupon bond $P(t_{i-1}, t_i)$ with strike X_i' and maturity t_{i-1}. Hence

$$\text{Fll}(t, t_{i-1}, t_i, \tau_i, N, X) = N_i'\,\text{ZBC}(t, t_{i-1}, t_i, X_i').$$

(d) Explain why caps are portfolios of zero-coupon bond puts, while floors are portfolios of zero-coupon bond calls.

A caplet pays when the future LIBOR rate is above the strike. Since a higher future LIBOR rate corresponds to a lower bond price $P(t_{i-1}, t_i)$, the caplet becomes a put on the corresponding zero-coupon bond. A floorlet pays when LIBOR is below the strike. Since a lower LIBOR rate corresponds to a higher bond price, the floorlet becomes a call on the zero-coupon bond. A cap is a sum of caplets, so it is a portfolio of bond puts; a floor is a sum of floorlets, so it is a portfolio of bond calls.

Problem A-3. Deferred payoffs

Let $t < \tau < T$, and suppose H_τ is known at time τ. Consider a claim that pays H_τ at the later date T.

(a) Starting from the risk-neutral pricing formula,

$$\pi_t = \mathbb{E}_t\left[e^{-\int_t^T r_s ds} H_\tau\right],$$

use the tower property of conditional expectation to show that

$$\pi_t = \mathbb{E}_t\left[e^{-\int_t^\tau r_s ds} H_\tau P(\tau, T)\right].$$

Start by splitting the discount factor:

$$e^{-\int_t^T r_s ds} = e^{-\int_t^\tau r_s ds} e^{-\int_\tau^T r_s ds}.$$

Then apply the tower property:

$$\begin{aligned}
\pi_t &= \mathbb{E}_t\left[e^{-\int_t^T r_s ds} H_\tau\right] \\
&= \mathbb{E}_t\left[\mathbb{E}_\tau\left(e^{-\int_t^T r_s ds} H_\tau\right)\right] \\
&= \mathbb{E}_t\left[e^{-\int_t^\tau r_s ds} H_\tau \mathbb{E}_\tau\left(e^{-\int_\tau^T r_s ds}\right)\right].
\end{aligned}$$

Since

$$P(\tau, T) = \mathbb{E}_\tau\left(e^{-\int_\tau^T r_s ds}\right),$$

we obtain

$$\pi_t = \mathbb{E}_t \left[e^{-\int_t^\tau r_s ds} H_\tau P(\tau, T) \right].$$

(b) Explain why this means that a payoff known at τ but paid at T may be treated as an anticipated payoff $H_\tau P(\tau, T)$ paid at τ.

At time τ, the amount H_τ is known, but it is not paid until T. The value at time τ of receiving H_τ at time T is therefore $H_\tau P(\tau, T)$. Thus, for pricing at the earlier time t, the original deferred payoff is equivalent to an anticipated payoff of $H_\tau P(\tau, T)$ at time τ.

(c) Explain how this idea helps interpret the caplet formulas from Problems A-1 and A-2.

In a caplet or floorlet, the rate is fixed at the reset date t_{i-1}, but the cash flow is paid at t_i. Once the rate is known at t_{i-1}, the later payment can be valued at t_{i-1} by multiplying by $P(t_{i-1}, t_i)$. This is exactly why the caplet and floorlet payoffs can be rewritten as options on the zero-coupon bond $P(t_{i-1}, t_i)$.

Problem B-1. Moving a payoff forward to a common terminal date

Let $t < T < S$, and let H be known at time T. Proposition 2.8.1 states that

$$\mathbb{E}_t[D(t, T)H] = \mathbb{E}_t \left[\frac{D(t, S)H}{P(T, S)} \right]. \tag{2.30}$$

(a) Prove this identity using $D(t, S) = D(t, T)D(T, S)$ and the tower property.

Using $D(t, S) = D(t, T)D(T, S)$,

$$\mathbb{E}_t \left[\frac{D(t, S)H}{P(T, S)} \right] = \mathbb{E}_t \left[\frac{D(t, T)D(T, S)H}{P(T, S)} \right].$$

Now condition on $\mathcal{F}_T$. Since $D(t, T)$, H, and $P(T, S)$ are known at time T,

$$\begin{aligned} \mathbb{E}_t \left[\frac{D(t, T)D(T, S)H}{P(T, S)} \right] &= \mathbb{E}_t \left[\mathbb{E}_T \left(\frac{D(t, T)D(T, S)H}{P(T, S)} \right) \right] \\ &= \mathbb{E}_t \left[\frac{D(t, T)H}{P(T, S)} \mathbb{E}_T(D(T, S)) \right]. \end{aligned}$$

But

$$\mathbb{E}_T(D(T, S)) = P(T, S).$$

Therefore

$$\mathbb{E}_t \left[\frac{D(t, S)H}{P(T, S)} \right] = \mathbb{E}_t[D(t, T)H],$$

which proves the identity.

(b) Give the financial interpretation of the factor $1/P(T, S)$.

The factor $1/P(T, S)$ is the forward accumulation factor from T to S. If H is known at time T, then receiving H at T is equivalent to receiving $H/P(T, S)$ at S, because the time-T

value of one unit paid at S is $P(T,S)$. Thus $1/P(T,S)$ converts a time-T amount into the equivalent time-S payoff.

(c) Explain why this identity is useful for pricing products with many payoff dates.

If a product has cash flows at many dates, pricing each cash flow under its own natural forward measure can become cumbersome, especially in Monte Carlo simulation. Formula (2.30) lets us move earlier cash flows forward to a common later date. This allows the whole product to be priced under one terminal forward measure rather than many different measures.

Problem B-2. Multiple cash flows and the terminal measure

Let $T_1 < \cdots < T_n$, and suppose an interest-rate derivative pays H_i at time T_i, where H_i is known at T_i. Assume there are no early-exercise features and $t < T_1$.

(a) Write the time-t price as a sum of discounted expectations:

$$\pi_t = \sum_{i=1}^{n} \mathbb{E}_t[D(t,T_i)H_i].$$

Since there are no early-exercise features, the derivative is simply the sum of its cash flows. By linearity of no-arbitrage pricing,

$$\pi_t = \sum_{i=1}^{n} \mathbb{E}_t[D(t,T_i)H_i].$$

Each term is the time-t value of the cash flow H_i paid at T_i.

(b) Rewrite the same price using the T_i-forward measure for each cash flow.

For each payoff date T_i, the T_i-forward measure gives

$$\mathbb{E}_t[D(t,T_i)H_i] = P(t,T_i)\mathbb{E}_t^{T_i}[H_i].$$

Therefore

$$\pi_t = \sum_{i=1}^{n} P(t,T_i)\mathbb{E}_t^{T_i}[H_i].$$

This is correct, but it uses a different forward measure for each payment date.

(c) Use Proposition 2.8.1 to rewrite all cash flows as if they were paid at the common terminal date T_n.

Apply (2.30) to each term with $T = T_i$ and $S = T_n$:

$$\mathbb{E}_t[D(t,T_i)H_i] = \mathbb{E}_t\left[\frac{D(t,T_n)H_i}{P(T_i,T_n)}\right].$$

Summing over i gives

$$\pi_t = \sum_{i=1}^{n} \mathbb{E}_t\left[\frac{D(t,T_n)H_i}{P(T_i,T_n)}\right].$$

By linearity,

$$\pi_t = \mathbb{E}_t \left[D(t, T_n) \sum_{i=1}^{n} \frac{H_i}{P(T_i, T_n)} \right].$$

(d) Deduce the terminal-measure representation

$$\pi_t = P(t, T_n) \mathbb{E}_t^{T_n} \left[\sum_{i=1}^{n} \frac{H_i}{P(T_i, T_n)} \right].$$

From part (c), the entire product can be viewed as a single payoff at T_n:

$$\sum_{i=1}^{n} \frac{H_i}{P(T_i, T_n)}.$$

Using the T_n-forward measure pricing formula,

$$\mathbb{E}_t \left[D(t, T_n) Y \right] = P(t, T_n) \mathbb{E}_t^{T_n} [Y],$$

with

$$Y = \sum_{i=1}^{n} \frac{H_i}{P(T_i, T_n)},$$

we obtain

$$\pi_t = P(t, T_n) \mathbb{E}_t^{T_n} \left[\sum_{i=1}^{n} \frac{H_i}{P(T_i, T_n)} \right].$$

(e) Explain why this form is especially convenient for Monte Carlo pricing.

Monte Carlo simulation is much easier when all state variables are simulated under one probability measure. The terminal-measure representation allows all cash flows to be valued under the single T_n-forward measure. Without this transformation, each cash flow would naturally live under a different forward measure, which would make simulation and measure changes more burdensome.

Problem B-3. Foreign payoff, domestic value

Let X_T^f be a payoff denominated in foreign currency at time T. Let B^f be the foreign money-market account, B the domestic money-market account, and Q_t the exchange rate, defined as domestic currency per one unit of foreign currency.

(a) Write the time-t value of X_T^f in the foreign market under the foreign risk-neutral measure $\mathbb{Q}^f$.

In the foreign market, the foreign-currency value is

$$V_t^f = B_t^f \mathbb{E}_t^f \left[\frac{X_T^f}{B_T^f} \right].$$

This is just the usual risk-neutral pricing formula, but using the foreign bank account and the foreign risk-neutral measure.

(b) Convert this foreign value into domestic currency at time t.

Since one unit of foreign currency is worth Q_t units of domestic currency, the domestic value of the foreign price is

$$V_t = Q_t V_t^f = Q_t B_t^f \mathbb{E}_t^f \left[\frac{X_T^f}{B_T^f} \right].$$

This is the left-hand side of (2.31).

(c) Now instead convert the future payoff into domestic currency at time T and price it under the domestic risk-neutral measure $\mathbb{Q}$.

At time T, the foreign payoff X_T^f is worth $Q_T X_T^f$ in domestic currency. Pricing that domestic payoff under the domestic risk-neutral measure gives

$$B_t \mathbb{E}_t \left[\frac{X_T^f Q_T}{B_T} \right].$$

This is the right-hand side of (2.31).

(d) Explain why no-arbitrage requires

$$Q_t B_t^f \mathbb{E}_t^f \left[\frac{X_T^f}{B_T^f} \right] = B_t \mathbb{E}_t \left[\frac{X_T^f Q_T}{B_T} \right]. \tag{2.31}$$

Both sides are domestic-currency values of the same economic claim. The left-hand side prices the claim in the foreign market and converts the resulting price into domestic currency today. The right-hand side converts the future payoff into domestic currency and prices it directly in the domestic market. If the two values differed, one could buy the cheaper representation and sell the more expensive representation, producing an arbitrage. Therefore no-arbitrage requires equality.

(e) In words, summarize the difference between the two sides of (2.31).

The left-hand side says: price foreign first, then convert today. The right-hand side says: convert the future payoff at maturity, then price domestically. Equation (2.31) says these two routes must lead to the same domestic value.

Problem B-4. The foreign-to-domestic density

Using equation (2.31) at time 0, derive the candidate Radon–Nikodym derivative

$$\frac{d\mathbb{Q}^f}{d\mathbb{Q}} = \frac{Q_T B_T^f}{Q_0 B_T}. \tag{2.32}$$

Then explain why this measure change can be viewed as a numeraire change.

At time 0, equation (2.31) becomes

$$Q_0 B_0^f \mathbb{E}^f \left[\frac{X_T^f}{B_T^f} \right] = B_0 \mathbb{E} \left[\frac{X_T^f Q_T}{B_T} \right].$$

With the usual normalization $B_0 = B_0^f = 1$, this becomes

$$Q_0 \mathbb{E}^f \left[\frac{X_T^f}{B_T^f} \right] = \mathbb{E} \left[\frac{X_T^f Q_T}{B_T} \right].$$

Equivalently,

$$\mathbb{E}^f \left[\frac{Q_0 X_T^f}{B_T^f} \right] = \mathbb{E} \left[\frac{X_T^f Q_T}{B_T} \right].$$

By the definition of the Radon–Nikodym derivative,

$$\mathbb{E}^f[Y] = \mathbb{E} \left[Y \frac{d\mathbb{Q}^f}{d\mathbb{Q}} \right].$$

Apply this with

$$Y = \frac{Q_0 X_T^f}{B_T^f}.$$

Then

$$\mathbb{E}^f \left[\frac{Q_0 X_T^f}{B_T^f} \right] = \mathbb{E} \left[\frac{Q_0 X_T^f}{B_T^f} \frac{d\mathbb{Q}^f}{d\mathbb{Q}} \right].$$

Comparing with

$$\mathbb{E} \left[\frac{X_T^f Q_T}{B_T} \right]$$

for arbitrary foreign payoff X_T^f gives

$$\frac{Q_0}{B_T^f} \frac{d\mathbb{Q}^f}{d\mathbb{Q}} = \frac{Q_T}{B_T}.$$

Rearranging yields

$$\frac{d\mathbb{Q}^f}{d\mathbb{Q}} = \frac{Q_T B_T^f}{Q_0 B_T}.$$

This is formula (2.32).

The same result can be read as a numeraire change. The foreign measure uses the foreign bank account B^f as numeraire. From a domestic perspective, the domestic bank account converted into foreign currency is B/Q. Thus moving between the foreign and domestic risk-neutral measures is equivalent to changing the unit of account from B^f to B/Q, with the exchange rate providing the conversion between currencies.