Interest Rate Models — Lecture 6 Notes

Sections 3.1–3.2.1: one-factor short-rate modeling and the Vasicek model

1 From term-structure objects to term-structure models

Chapters 1 and 2 introduced the objects of the interest-rate market: zero-coupon bonds, spot rates, forward rates, swap rates, no-arbitrage pricing, and numeraire change. Chapter 3 begins the modeling part of the course. The short-rate approach starts by choosing stochastic dynamics for the instantaneous short rate $r(t)$. Then bond prices and rates are derived from that process:

$$\boxed{P(t,T) = E_t^Q\left[\exp\left(-\int_t^T r(s)\,ds\right)\right].} \tag{3.2}$$

Thus the modeling chain is

$$\text{choose dynamics for } r(t) \quad\Longrightarrow\quad \text{compute } P(t,T) \quad\Longrightarrow\quad \text{derive rates and derivative prices.}$$

A one-factor short-rate model uses one Brownian driver for $r(t)$. Since the whole curve is then recovered from $r(t)$, the curve is ultimately generated by one source of uncertainty. This gives tractability, but it cannot fully describe shifts, steepening, flattening, and twists of the yield curve.

For pricing we work under the risk-neutral measure Q, not necessarily the objective measure Q^0. The bridge between the two is the market price of risk, which changes the drift when moving between the real-world and pricing descriptions. For derivative pricing alone, the model is often specified directly under Q; historical estimation instead concerns dynamics under Q^0.

Classical time-homogeneous short-rate models are endogenous term-structure models: the initial curve is generated by model parameters, so generally

$$P^{\text{model}}(0,T) \neq P^{\text{market}}(0,T).$$

Later exogenous extensions take the market curve as an input and fit it exactly. For example, a time-dependent mean-reversion level in Vasicek leads toward Hull-White-type models.

2 Model-selection questions

For each short-rate model, keep the following questions in mind:

- Does the model imply positive rates, or can $r(t)$ become negative?
- What distribution does it imply for $r(t)$?
- Are zero-coupon bond prices $P(t,T)$ explicit?
- Are bond-option, cap, floor, or swaption prices explicit?
- Is the model mean reverting, calibratable, and usable for Monte Carlo or lattice pricing?

The classical path begins with Vasicek, Dothan, CIR, and Exponential Vasicek; later sections move to Hull-White, Black-Karasinski, CIR++, and related extensions. Vasicek is the cleanest starting point because it is Gaussian, mean-reverting, and analytically tractable.

3 The Vasicek model under the risk-neutral measure

The Vasicek model assumes that the short rate follows an Ornstein-Uhlenbeck process with constant coefficients under Q:

$$\boxed{dr(t) = k[\theta - r(t)]dt + \sigma dW(t), \qquad r(0) = r_0.} \tag{3.5}$$

Here r_0 is the initial short rate, $k > 0$ is the mean-reversion speed, $\theta > 0$ is the long-run mean level, and $\sigma > 0$ is the absolute volatility. The drift is positive when $r(t) < \theta$ and negative when $r(t) > \theta$, so the process is pulled toward θ.

Solving the SDE gives, for $s \le t$,

$$r(t) = r(s)e^{-k(t-s)} + \theta\big(1 - e^{-k(t-s)}\big) + \sigma\int_s^t e^{-k(t-u)}\,dW(u). \tag{3.6}$$

This is

$$\text{fading old rate} \;+\; \text{long-run mean pull} \;+\; \text{fading random shocks.}$$

Conditional on $\mathcal{F}_s$, the short rate is normally distributed with

$$E\{r(t) \mid \mathcal{F}_s\} = r(s)e^{-k(t-s)} + \theta\big(1 - e^{-k(t-s)}\big),$$

$$\operatorname{Var}\{r(t) \mid \mathcal{F}_s\} = \frac{\sigma^2}{2k}\big(1 - e^{-2k(t-s)}\big). \tag{3.7}$$

The Gaussian structure is the source of Vasicek's tractability. It is also the source of its main defect: $r(t)$ can be negative with positive probability.

4 Bond prices and the affine term structure

In the Vasicek model, the zero-coupon bond price is affine in the short rate:

$$\boxed{P(t,T) = A(t,T)e^{-B(t,T)r(t)}.} \tag{3.8}$$

where

$$B(t,T) = \frac{1}{k}\big(1 - e^{-k(T-t)}\big),$$
$$A(t,T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4k}B(t,T)^2\right\}.$$

Taking logs gives

$$\ln P(t,T) = \ln A(t,T) - B(t,T)r(t),$$

so the log bond price is a deterministic constant plus a deterministic coefficient times the state variable. This is the affine term-structure property. The term

$$B(t,T) = \int_t^T e^{-k(u-t)}\,du$$

comes from the fading effect of today's short rate inside the integrated short-rate path; the remaining deterministic terms enter $A(t,T)$.

5 Forward-measure dynamics and bond options

For a fixed maturity T, changing from the money-market numeraire to the T-bond numeraire changes the drift. Under the T-forward measure Q^T,

$$dr(t) = \big[k\theta - B(t,T)\sigma^2 - kr(t)\big]dt + \sigma dW^T(t), \tag{3.9}$$

where

$$dW^T(t) = dW(t) + \sigma B(t,T)dt.$$

Volatility is unchanged; only the drift is shifted by the numeraire-change correction $-B(t,T)\sigma^2$. The distribution remains Gaussian under Q^T, but its mean changes.

For a European option with maturity T, strike X, written on a zero-coupon bond maturing at $S > T$,

$$ZBO(t,T,S,X) = \omega\Big[P(t,S)\Phi(\omega h) - XP(t,T)\Phi\big(\omega(h-\sigma_p)\big)\Big], \tag{3.10}$$

where $\omega = 1$ for a call and $\omega = -1$ for a put,

$$\sigma_p = \sigma\sqrt{\frac{1-e^{-2k(T-t)}}{2k}}\,B(T,S), \qquad h = \frac{1}{\sigma_p}\ln\!\left(\frac{P(t,S)}{P(t,T)X}\right) + \frac{\sigma_p}{2}.$$

This is a Black-like formula for bond options: h plays a d_1-type role and $h - \sigma_p$ plays the companion d_2 role.

6 Objective-measure dynamics and AR(1) estimation

For real-world estimation, Vasicek is written under the objective measure Q^0. Brigo and Mercurio use a tractable market-price-of-risk specification that keeps the model linear and Gaussian under both measures. A convenient objective-measure form is

$$dr(t) = [b - ar(t)]dt + \sigma dW^0(t). \tag{3.12}$$

Solving between s and t gives

$$r(t) = r(s)e^{-a(t-s)} + \frac{b}{a}\big(1 - e^{-a(t-s)}\big) + \sigma\int_s^t e^{-a(t-u)}\,dW^0(u). \tag{3.13}$$

If observations are spaced by a time step δ, this becomes an AR(1)-type transition:

$$r_i = \beta(1-\alpha) + \alpha r_{i-1} + \varepsilon_i, \qquad \beta = \frac{b}{a}, \quad \alpha = e^{-a\delta}, \quad V^2 = \frac{\sigma^2}{2a}\big(1 - e^{-2a\delta}\big). \tag{3.14}$$

Here α is the persistence from one observation to the next, β is the long-run mean under Q^0, and V^2 is the one-step conditional variance. These three quantities determine the discrete-time transition probability of the observed short-rate proxy.

Summary

Vasicek is the first short-rate laboratory: choose Gaussian mean-reverting dynamics for $r(t)$, derive affine bond prices $P(t,T) = A(t,T)e^{-B(t,T)r(t)}$, and use numeraire change for forward-measure pricing. Its strength is tractability; its weakness is that Gaussian rates can be negative. Under Q^0, the discretized transition is AR(1)-type, linking the continuous-time model to historical estimation.

Next lecture: Dothan, CIR, and the first comparison of short-rate-model tradeoffs.