Interest Rate Models: Theory and Practice

Problem Set 6

Part A

Part A concerns the short-rate modeling philosophy, the Vasicek short-rate dynamics, and the affine bond-pricing formula.

Problem A-1. From term-structure objects to short-rate dynamics

In the first five lectures, the basic objects were bond prices, spot rates, forward rates, swap rates, and pricing measures. Chapter 3 begins by choosing stochastic dynamics for the short rate.

(a) Explain the short-rate modeling pipeline

$$\text{choose dynamics for } r(t) \quad \Longrightarrow \quad \text{compute } P(t,T) \quad \Longrightarrow \quad \text{derive rates and derivative prices.}$$

(b) State the risk-neutral bond-pricing identity for a zero-coupon bond paying one unit at time T.

(c) Explain what makes a model "one-factor" in this setting.

(d) Explain the distinction between an endogenous initial term structure and an exogenous, exactly fitted initial term structure.

Problem A-2. Solving the Vasicek short-rate dynamics

Under the risk-neutral measure $\mathbb{Q}$, the Vasicek model is

$$\mathrm{d}r(t) = k[\theta - r(t)]\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \qquad r(0) = r_0,$$

where r_0, k, θ, σ are positive constants.

(a) Derive the integrated solution, for $s \le t$,

$$r(t) = r(s)e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-u)}\,\mathrm{d}W(u).$$

(b) Compute $\mathbb{E}[r(t) \mid \mathcal{F}_s]$ and $\mathrm{Var}[r(t) \mid \mathcal{F}_s]$.

(c) Explain why the model is mean-reverting and identify the long-run mean.

(d) Explain why the Vasicek short rate can be negative with positive probability.

Problem A-3. The affine bond-pricing form

In the Vasicek model, the zero-coupon bond price has the form

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

where

$$B(t,T) = \frac{1 - e^{-k(T-t)}}{k}$$

and

$$A(t,T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4k}B(t,T)^2\right\}.$$

(a) Take logs and show that the model is affine in the state variable $r(t)$.

(b) Interpret the role of $B(t,T)$ as the loading of the bond price on the current short rate.

(c) Explain why the Gaussian structure of Vasicek makes the expectation

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,\mathrm{d}s\right)\right]$$

analytically tractable.

(d) Explain why this classical time-homogeneous model may fail to reproduce the market curve $P^M(0,T)$ exactly.

Part B

Part B concerns the T-forward measure, zero-coupon bond options, and the link between objective-measure estimation and risk-neutral pricing.

Problem B-1. Vasicek dynamics under the T-forward measure

For a fixed maturity T, the T-forward measure is associated with the numeraire $P(\cdot,T)$. In the Vasicek model,

$$\mathrm{d}W^T(t) = \mathrm{d}W(t) + \sigma B(t,T)\,\mathrm{d}t.$$

(a) Starting from the risk-neutral dynamics, derive the T-forward-measure dynamics

$$\mathrm{d}r(t) = \left[k\theta - \sigma^2 B(t,T) - kr(t)\right]\,\mathrm{d}t + \sigma\,\mathrm{d}W^T(t).$$

(b) Identify exactly which part of the drift changes when moving from $\mathbb{Q}$ to $\mathbb{Q}^T$.

(c) Explain why the diffusion coefficient and the conditional variance of $r(t)$ do not change under this measure change.

(d) Relate this calculation to the numeraire-change toolkit from Chapter 2.

Problem B-2. Bond options in the Vasicek model

Let $t < T < S$. Consider a European option with maturity T, strike X, and underlying zero-coupon bond $P(T,S)$. The Vasicek bond-option formula can be written as

$$ZBO(t,T,S,X) = \omega\left[P(t,S)\Phi(\omega h) - XP(t,T)\Phi(\omega(h-\sigma_p))\right],$$

where $\omega = 1$ for a call and $\omega = -1$ for a put,

$$\sigma_p = \sigma\sqrt{\frac{1-e^{-2k(T-t)}}{2k}}\,B(T,S), \qquad h = \frac{1}{\sigma_p}\log\left(\frac{P(t,S)}{P(t,T)X}\right) + \frac{\sigma_p}{2}.$$

(a) Explain why the natural pricing measure for this option is the T-forward measure.

(b) Explain the role of σ_p in the formula.

(c) Explain why h plays the same structural role as a Black-Scholes d_1 term.

(d) Use put-call parity for bond options to write the corresponding put price in terms of the call price.

Problem B-3. Objective-measure dynamics and the market price of risk

Brigo and Mercurio write the objective-measure Vasicek dynamics as

$$\mathrm{d}r(t) = [k\theta - (k + \lambda\sigma)r(t)]\,\mathrm{d}t + \sigma\,\mathrm{d}W^0(t).$$

(a) Compare this equation with the risk-neutral Vasicek dynamics.

(b) What happens when $\lambda = 0$?

(c) Explain why the market price of risk acts as a bridge between $\mathbb{Q}_0$ and $\mathbb{Q}$.

(d) Explain why historical estimation concerns the $\mathbb{Q}_0$ dynamics, while derivative pricing concerns the $\mathbb{Q}$ dynamics.

Problem B-4. The AR(1) transition implied by Vasicek

Rewrite the objective-measure dynamics as

$$\mathrm{d}r(t) = [b - ar(t)]\,\mathrm{d}t + \sigma\,\mathrm{d}W^0(t).$$

Suppose observations $r_0, r_1, \ldots, r_n$ are spaced by a fixed time step δ.

(a) Integrate the SDE over one time step to show that

$$r_i = \beta(1 - \alpha) + \alpha r_{i-1} + \varepsilon_i,$$

where

$$\beta = \frac{b}{a}, \qquad \alpha = e^{-a\delta}, \qquad \varepsilon_i \sim N(0, V^2).$$

(b) Show that

$$V^2 = \frac{\sigma^2}{2a}\left(1 - e^{-2a\delta}\right).$$

(c) Interpret α, β, and V^2 financially or statistically.

(d) Explain why the AR(1) representation gives complete information about the one-step transition distribution of r under $\mathbb{Q}_0$.