

# Interest Rate Models: Theory and Practice

## Solutions to Problem Set 6

### Problem A-1. From term-structure objects to short-rate dynamics

In the first five lectures, the basic objects were bond prices, spot rates, forward rates, swap rates, and pricing measures. Chapter 3 begins by choosing stochastic dynamics for the short rate.

(a) Explain the short-rate modeling pipeline

choose dynamics for  $r(t)$   $\implies$  compute  $P(t, T)$   $\implies$  derive rates and derivative prices.

(b) State the risk-neutral bond-pricing identity for a zero-coupon bond paying one unit at time  $T$ .

(c) Explain what makes a model “one-factor” in this setting.

(d) Explain the distinction between an endogenous initial term structure and an exogenous, exactly fitted initial term structure.

*Solution.*

(a) The short-rate approach starts by specifying the stochastic law of motion for the instantaneous spot rate  $r(t)$ . Once the future path of  $r$  is modeled, the model can compute stochastic discount factors and therefore zero-coupon bond prices. After bond prices are known, spot rates, forward rates, swap rates, caps, floors, and bond options can be derived from those bond prices. Thus the model does not start from every point on the curve separately. It starts from the state variable  $r(t)$  and generates the curve from that state variable.

(b) The risk-neutral bond-pricing identity is

$$P(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[ \exp \left( - \int_t^T r(s) ds \right) \right].$$

This is the special case of the no-arbitrage pricing formula for a claim paying one unit at time  $T$ .

(c) A model is one-factor when one Brownian motion, or one scalar source of randomness, drives the short-rate process. Since the short rate is then used to generate the whole term structure, the entire yield curve is ultimately being driven by one source of uncertainty.

(d) In an endogenous term-structure model, the initial curve is generated internally by the model parameters. Therefore the model-implied curve  $P^{\text{model}}(0, T)$  need not equal the observed market curve  $P^M(0, T)$  for all maturities. In an exogenous or exactly fitted model, the current market curve is taken as an input and the model is shifted or extended so that it reproduces that initial curve exactly.

### Problem A-2. Solving the Vasicek short-rate dynamics

Under the risk-neutral measure  $\mathbb{Q}$ , the Vasicek model is

$$dr(t) = k[\theta - r(t)] dt + \sigma dW(t), \quad r(0) = r_0,$$

where  $r_0, k, \theta, \sigma$  are positive constants.

(a) Derive the integrated solution, for  $s \leq t$ ,

$$r(t) = r(s)e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-u)} dW(u).$$

- (b) Compute  $\mathbb{E}[r(t) | \mathcal{F}_s]$  and  $\text{Var}[r(t) | \mathcal{F}_s]$ .
- (c) Explain why the model is mean-reverting and identify the long-run mean.
- (d) Explain why the Vasicek short rate can be negative with positive probability.

*Solution.*

- (a) Rewrite the SDE as

$$dr(t) + kr(t) dt = k\theta dt + \sigma dW(t).$$

Multiply by the integrating factor  $e^{kt}$ :

$$d(e^{kt}r(t)) = k\theta e^{kt} dt + \sigma e^{kt} dW(t).$$

Integrating from  $s$  to  $t$  gives

$$e^{kt}r(t) - e^{ks}r(s) = \theta(e^{kt} - e^{ks}) + \sigma \int_s^t e^{ku} dW(u).$$

Dividing by  $e^{kt}$  yields

$$r(t) = r(s)e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-u)} dW(u).$$

- (b) Conditional on  $\mathcal{F}_s$ , the stochastic integral has mean zero. Therefore

$$\mathbb{E}[r(t) | \mathcal{F}_s] = r(s)e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}).$$

By Ito isometry,

$$\text{Var}[r(t) | \mathcal{F}_s] = \sigma^2 \int_s^t e^{-2k(t-u)} du = \frac{\sigma^2}{2k} (1 - e^{-2k(t-s)}).$$

- (c) The drift is  $k(\theta - r(t))$ . If  $r(t) < \theta$ , the drift is positive and pushes the short rate upward on average. If  $r(t) > \theta$ , the drift is negative and pushes the short rate downward on average. Also,

$$\mathbb{E}[r(t) | \mathcal{F}_s] \rightarrow \theta \quad \text{as } t \rightarrow \infty,$$

so  $\theta$  is the long-run mean under the risk-neutral dynamics.

- (d) The conditional distribution of  $r(t)$  is normal. A normal random variable has support on the entire real line unless its variance is zero. Since the variance is positive for  $t > s$  when  $\sigma > 0$ , the Vasicek model assigns positive probability to negative short rates.

### Problem A-3. The affine bond-pricing form

In the Vasicek model, the zero-coupon bond price has the form

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)},$$

where

$$B(t, T) = \frac{1 - e^{-k(T-t)}}{k}$$

and

$$A(t, T) = \exp \left\{ \left( \theta - \frac{\sigma^2}{2k^2} \right) [B(t, T) - T + t] - \frac{\sigma^2}{4k} B(t, T)^2 \right\}.$$

- (a) Take logs and show that the model is affine in the state variable  $r(t)$ .
- (b) Interpret the role of  $B(t, T)$  as the loading of the bond price on the current short rate.
- (c) Explain why the Gaussian structure of Vasicek makes the expectation

$$P(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[ \exp \left( - \int_t^T r(s) ds \right) \right]$$

analytically tractable.

- (d) Explain why this classical time-homogeneous model may fail to reproduce the market curve  $P^M(0, T)$  exactly.

*Solution.*

- (a) Taking logarithms gives

$$\log P(t, T) = \log A(t, T) - B(t, T)r(t).$$

This is affine in  $r(t)$  because it has the form

$$\text{constant in } r(t) + \text{coefficient} \times r(t).$$

The constant term is  $\log A(t, T)$  and the coefficient on the state variable is  $-B(t, T)$ .

- (b) The factor  $B(t, T)$  measures how sensitive the log bond price is to the current short rate:

$$\frac{\partial \log P(t, T)}{\partial r(t)} = -B(t, T).$$

For longer maturities,  $B(t, T)$  is typically larger, so the current short rate has a larger effect on long-maturity bond prices than on very short-maturity bond prices.

- (c) Under Vasicek,  $r$  is Gaussian and linear. Therefore the integral  $\int_t^T r(s) ds$  is also Gaussian because it is a linear functional of a Gaussian process. The bond price requires the expectation of the exponential of a Gaussian random variable, and that expectation is available in closed form. This is the source of the tractability behind the affine formula.
- (d) The model has only a few constant parameters, namely  $r_0, k, \theta, \sigma$ . These parameters generate the initial model curve internally. A real market curve may have shapes that cannot be produced exactly by such a small time-homogeneous parameter set. Therefore  $P^{\text{model}}(0, T)$  need not match  $P^M(0, T)$  for every  $T$ .

**Problem B-1. Vasicek dynamics under the  $T$ -forward measure**

For a fixed maturity  $T$ , the  $T$ -forward measure is associated with the numeraire  $P(\cdot, T)$ . In the Vasicek model,

$$dW^T(t) = dW(t) + \sigma B(t, T) dt.$$

- (a) Starting from the risk-neutral dynamics, derive the  $T$ -forward-measure dynamics

$$dr(t) = \left[ k\theta - \sigma^2 B(t, T) - kr(t) \right] dt + \sigma dW^T(t).$$

- (b) Identify exactly which part of the drift changes when moving from  $\mathbb{Q}$  to  $\mathbb{Q}^T$ .
- (c) Explain why the diffusion coefficient and the conditional variance of  $r(t)$  do not change under this measure change.

(d) Relate this calculation to the numeraire-change toolkit from Chapter 2.

*Solution.*

(a) Under  $\mathbb{Q}$ ,

$$dr(t) = (k\theta - kr(t)) dt + \sigma dW(t).$$

The Brownian motions are related by

$$dW^T(t) = dW(t) + \sigma B(t, T) dt,$$

so

$$dW(t) = dW^T(t) - \sigma B(t, T) dt.$$

Substitute this into the risk-neutral dynamics:

$$dr(t) = (k\theta - kr(t)) dt + \sigma [dW^T(t) - \sigma B(t, T) dt].$$

Therefore

$$dr(t) = [k\theta - \sigma^2 B(t, T) - kr(t)] dt + \sigma dW^T(t).$$

(b) The original drift is  $k\theta - kr(t)$ . Under  $\mathbb{Q}^T$ , the drift becomes

$$k\theta - kr(t) - \sigma^2 B(t, T).$$

Thus the measure change subtracts the deterministic correction  $\sigma^2 B(t, T)$  from the drift.

(c) The change of measure recenters the Brownian motion, so it changes the drift. It does not change the instantaneous diffusion loading  $\sigma$ . Since the conditional variance of the Ornstein-Uhlenbeck transition depends on the diffusion coefficient and the mean-reversion speed, not on the deterministic drift level, the conditional variance remains

$$\frac{\sigma^2}{2k} (1 - e^{-2k(t-s)}).$$

(d) This is exactly the Chapter 2 idea. Changing from the bank-account numeraire to the  $T$ -bond numeraire changes the associated measure. The price is invariant, but the Brownian shock must be recentered so that prices normalized by  $P(\cdot, T)$  are martingales under  $\mathbb{Q}^T$ . That recentering appears as a drift correction in the short-rate dynamics.

### Problem B-2. Bond options in the Vasicek model

Let  $t < T < S$ . Consider a European option with maturity  $T$ , strike  $X$ , and underlying zero-coupon bond  $P(T, S)$ . The Vasicek bond-option formula can be written as

$$ZBO(t, T, S, X) = \omega [P(t, S)\Phi(\omega h) - XP(t, T)\Phi(\omega(h - \sigma_p))],$$

where  $\omega = 1$  for a call and  $\omega = -1$  for a put,

$$\sigma_p = \sigma \sqrt{\frac{1 - e^{-2k(T-t)}}{2k}} B(T, S), \quad h = \frac{1}{\sigma_p} \log \left( \frac{P(t, S)}{P(t, T)X} \right) + \frac{\sigma_p}{2}.$$

(a) Explain why the natural pricing measure for this option is the  $T$ -forward measure.

(b) Explain the role of  $\sigma_p$  in the formula.

- (c) Explain why  $h$  plays the same structural role as a Black-Scholes  $d_1$  term.
- (d) Use put-call parity for bond options to write the corresponding put price in terms of the call price.

*Solution.*

- (a) The option payoff is realized at time  $T$ . Therefore the natural numeraire is the  $T$ -maturity zero-coupon bond  $P(\cdot, T)$ , whose terminal value is  $P(T, T) = 1$ . Under the  $T$ -forward measure,

$$\text{Price}_t = P(t, T) \mathbb{E}_t^T[\text{payoff at } T],$$

so the stochastic discount factor is absorbed into the measure change.

- (b) The quantity  $\sigma_p$  is the effective standard deviation of the log bond price over the option horizon. Since

$$P(T, S) = A(T, S) e^{-B(T, S)r(T)},$$

uncertainty in  $P(T, S)$  comes from uncertainty in  $r(T)$  under the  $T$ -forward measure. The term  $B(T, S)$  converts short-rate uncertainty into bond-price uncertainty, while

$$\sigma \sqrt{\frac{1 - e^{-2k(T-t)}}{2k}}$$

is the conditional standard deviation of  $r(T)$ .

- (c) The term  $h$  has the form

$$\frac{\log(\text{forward}/\text{strike})}{\text{volatility}} + \frac{1}{2} \text{volatility}.$$

This is the same structure as the Black-Scholes  $d_1$  term. Here the relevant forward object is  $P(t, S)/P(t, T)$ , the time- $t$  forward price of the  $S$ -maturity bond for delivery at  $T$ , and the strike is  $X$ .

- (d) Put-call parity for bond options is

$$ZBC(t, T, S, X) + XP(t, T) = ZBP(t, T, S, X) + P(t, S).$$

Therefore

$$ZBP(t, T, S, X) = ZBC(t, T, S, X) - P(t, S) + XP(t, T).$$

### Problem B-3. Objective-measure dynamics and the market price of risk

Brigo and Mercurio write the objective-measure Vasicek dynamics as

$$dr(t) = [k\theta - (k + \lambda\sigma)r(t)] dt + \sigma dW^0(t).$$

- (a) Compare this equation with the risk-neutral Vasicek dynamics.
- (b) What happens when  $\lambda = 0$ ?
- (c) Explain why the market price of risk acts as a bridge between  $\mathbb{Q}_0$  and  $\mathbb{Q}$ .
- (d) Explain why historical estimation concerns the  $\mathbb{Q}_0$  dynamics, while derivative pricing concerns the  $\mathbb{Q}$  dynamics.

*Solution.*

(a) The risk-neutral dynamics are

$$dr(t) = [k\theta - kr(t)] dt + \sigma dW(t).$$

The objective dynamics are

$$dr(t) = [k\theta - (k + \lambda\sigma)r(t)] dt + \sigma dW^0(t).$$

The diffusion coefficient is the same, but the drift has changed. Under  $\mathbb{Q}_0$ , the coefficient on  $r(t)$  is  $-(k + \lambda\sigma)$  rather than  $-k$ .

- (b) If  $\lambda = 0$ , then the objective and risk-neutral drift coefficients coincide. In that case, there is no market-price-of-risk wedge between the two dynamics, and the two SDEs have the same functional form and the same drift parameters.
- (c) The market price of risk determines the Girsanov change of measure that moves between the physical world and the pricing world. It tells us how much the drift must be adjusted when the Brownian motion is recentered from  $\mathbb{Q}_0$  to  $\mathbb{Q}$ . In this Vasicek specification, the market price of risk is chosen so that the model remains linear and Gaussian under both measures.
- (d) Historical observations of interest rates are generated in the real world, so their time-series distribution is a  $\mathbb{Q}_0$  distribution. Maximum likelihood or regression-based estimation therefore estimates objective-measure transition dynamics. Derivative prices, however, are no-arbitrage expectations under the risk-neutral measure  $\mathbb{Q}$ . Thus pricing and calibration to derivative prices must use the  $\mathbb{Q}$  dynamics, not the historical drift directly.

**Problem B-4. The AR(1) transition implied by Vasicek**

Rewrite the objective-measure dynamics as

$$dr(t) = [b - ar(t)] dt + \sigma dW^0(t).$$

Suppose observations  $r_0, r_1, \dots, r_n$  are spaced by a fixed time step  $\delta$ .

(a) Integrate the SDE over one time step to show that

$$r_i = \beta(1 - \alpha) + \alpha r_{i-1} + \varepsilon_i,$$

where

$$\beta = \frac{b}{a}, \quad \alpha = e^{-a\delta}, \quad \varepsilon_i \sim N(0, V^2).$$

(b) Show that

$$V^2 = \frac{\sigma^2}{2a} (1 - e^{-2a\delta}).$$

- (c) Interpret  $\alpha$ ,  $\beta$ , and  $V^2$  financially or statistically.
- (d) Explain why the AR(1) representation gives complete information about the one-step transition distribution of  $r$  under  $\mathbb{Q}_0$ .

*Solution.*

(a) The integrated solution of

$$dr(t) = [b - ar(t)] dt + \sigma dW^0(t)$$

over the interval  $[t_{i-1}, t_i]$ , where  $t_i - t_{i-1} = \delta$ , is

$$r_i = r_{i-1}e^{-a\delta} + \frac{b}{a}(1 - e^{-a\delta}) + \sigma \int_{t_{i-1}}^{t_i} e^{-a(t_i-u)} dW^0(u).$$

Define

$$\alpha = e^{-a\delta}, \quad \beta = \frac{b}{a},$$

and

$$\varepsilon_i = \sigma \int_{t_{i-1}}^{t_i} e^{-a(t_i-u)} dW^0(u).$$

Then

$$r_i = \beta(1 - \alpha) + \alpha r_{i-1} + \varepsilon_i.$$

(b) By Ito isometry,

$$\text{Var}(\varepsilon_i) = \sigma^2 \int_{t_{i-1}}^{t_i} e^{-2a(t_i-u)} du.$$

Let  $v = t_i - u$ . Then as  $u$  moves from  $t_{i-1}$  to  $t_i$ ,  $v$  moves from  $\delta$  to 0. Hence

$$\text{Var}(\varepsilon_i) = \sigma^2 \int_0^\delta e^{-2av} dv = \frac{\sigma^2}{2a}(1 - e^{-2a\delta}).$$

Therefore

$$V^2 = \frac{\sigma^2}{2a}(1 - e^{-2a\delta}).$$

- (c) The parameter  $\alpha = e^{-a\delta}$  is the persistence from one observation to the next. If  $\alpha$  is close to one, shocks fade slowly and mean reversion is weak over one time step. The parameter  $\beta = b/a$  is the long-run mean under the objective dynamics. The quantity  $V^2$  is the one-step conditional innovation variance, measuring the uncertainty in  $r_i$  after conditioning on  $r_{i-1}$ .
- (d) The AR(1) representation says that conditional on  $r_{i-1}$ ,

$$r_i | r_{i-1} \sim N\left(\beta(1 - \alpha) + \alpha r_{i-1}, V^2\right).$$

A normal transition distribution is completely determined by its conditional mean and conditional variance. Therefore the three quantities  $\alpha$ ,  $\beta$ , and  $V^2$  give complete information about the one-step transition law under  $\mathbb{Q}_0$ .