Interest Rate Models — Lecture 7 Notes

Sections 3.2.2–3.2.5: Dothan, CIR, affine term-structure models, and Exponential Vasicek

1 From Vasicek to positive-rate models

The Vasicek model is Gaussian, mean-reverting, and analytically tractable, but it assigns positive probability to negative short rates. The next classical models ask whether one can keep useful tractability while improving the behavior of the short rate.

The model-selection questions remain the same:

- Does the model keep rates positive?
- Is the short rate mean-reverting toward a meaningful long-run level?
- Is the distribution of $r(t)$ known?
- Are zero-coupon bond prices and bond options explicit?
- Is the resulting term structure affine?

The four models in this lecture should be read as different tradeoffs among positivity, mean reversion, and tractability.

2 The Dothan model

The Dothan model assumes that, under the risk-neutral measure,

$$dr(t) = ar(t)dt + \sigma r(t)dW(t), \qquad r(0) = r_0. \tag{3.17}$$

This is a geometric Brownian motion for the short rate. Hence, for $s \le t$,

$$r(t) = r(s) \exp\left\{ \left(a - \frac{1}{2}\sigma^2 \right)(t-s) + \sigma[W(t) - W(s)] \right\}. \tag{3.18}$$

Therefore $r(t)$ is conditionally lognormal and strictly positive. Its conditional moments are

$$\mathbb{E}[r(t) \mid \mathcal{F}_s] = r(s)e^{a(t-s)}, \tag{1}$$

$$\mathrm{Var}[r(t) \mid \mathcal{F}_s] = r(s)^2 e^{2a(t-s)} \left(e^{\sigma^2(t-s)} - 1 \right). \tag{3.19}$$

The gain is positivity. The cost is weak mean reversion. If $a < 0$, the mean tends to zero, not to a positive long-run level. If $a \ge 0$, the mean does not revert downward.

Dothan is unusual because it gives an analytical formula for pure discount bonds, but the expression involves special functions and a double numerical integral. Thus the formula is explicit in principle but not very practical. No analytical formula for a zero-coupon bond option is available.

There is also a deeper warning. Since r is lognormal, expectations involving the bank account can involve objects of the form

$$\mathbb{E}\{\exp(\exp(Y))\},$$

where Y is normally distributed. Such expectations may be infinite. This is the bank-account explosion problem for lognormal short-rate models.

3 The Cox-Ingersoll-Ross model

The CIR model changes the volatility term rather than making the whole short rate lognormal:

$$dr(t) = k[\theta - r(t)]dt + \sigma\sqrt{r(t)}dW(t), \qquad r(0) = r_0. \tag{3.21}$$

Here $k > 0$ is the mean-reversion speed, $\theta > 0$ is the long-run level, and $\sigma > 0$ controls volatility. The square-root diffusion coefficient makes volatility smaller when rates are close to zero.

The condition

$$2k\theta > \sigma^2 \tag{2}$$

ensures that the origin is inaccessible, so the short rate remains positive.

The CIR short rate has a noncentral chi-squared distribution. Its conditional mean and variance are

$$\mathbb{E}[r(t) \mid \mathcal{F}_s] = r(s)e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}), \tag{3}$$

$$\mathrm{Var}[r(t) \mid \mathcal{F}_s] = r(s)\frac{\sigma^2}{k}\left(e^{-k(t-s)} - e^{-2k(t-s)}\right) + \theta\frac{\sigma^2}{2k}\left(1 - e^{-k(t-s)}\right)^2. \tag{3.23}$$

The mean has the same structure as Vasicek: fading old rate plus long-run mean pull. The variance is state-dependent because the diffusion coefficient depends on $\sqrt{r(t)}$.

CIR also has affine zero-coupon bond prices:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}. \tag{3.24}$$

This is the main reason CIR is so important: it keeps positivity and mean reversion while preserving an affine bond-price structure.

However, the state-dependent volatility makes some option-pricing and forward-rate dynamics less clean than in Gaussian models. Under the S-forward measure, the forward rate dynamics implied by CIR are not the simple lognormal form

$$dF(t;T,S) = \sigma(t)F(t;T,S)dW^S(t).$$

Thus CIR is tractable, but not as algebraically simple as Vasicek.

4 Affine term-structure models

A model has an affine term structure if the continuously compounded spot rate is affine in the short rate:

$$R(t,T) = \alpha(t,T) + \beta(t,T)r(t). \tag{4}$$

This occurs whenever bond prices have the form

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}. \tag{5}$$

Indeed,

$$R(t,T) = -\frac{\ln A(t,T)}{T-t} + \frac{B(t,T)}{T-t}r(t). \tag{6}$$

Vasicek and CIR are affine models. Dothan is not affine.

For a general short-rate model

$$dr(t) = b(t,r(t))dt + \sigma(t,r(t))dW(t),$$

the model has an affine term structure when both the drift and instantaneous variance are affine in the state variable:

$$b(t,x) = \lambda(t)x + \eta(t), \qquad \sigma^2(t,x) = \gamma(t)x + \delta(t). \tag{7}$$

Then the functions A and B are obtained from Riccati-type differential equations. In Vasicek, $\gamma = 0$ and $\delta = \sigma^2$. In CIR, $\gamma = \sigma^2$ and $\delta = 0$.

The instantaneous forward-rate volatility in an affine model is

$$\sigma_f(t,T) = \frac{\partial B(t,T)}{\partial T}\sigma(t, r(t)). \tag{3.29}$$

So affine structure does not only produce bond prices; it also determines the volatility structure of the forward curve.

5 The Exponential-Vasicek model

The Exponential-Vasicek model makes the logarithm of the short rate follow an Ornstein-Uhlenbeck process:

$$dy(t) = [\theta - ay(t)]dt + \sigma dW(t), \qquad r(t) = e^{y(t)}. \tag{8}$$

Applying Ito's formula gives

$$dr(t) = r(t)\left[\theta + \frac{\sigma^2}{2} - a\ln r(t)\right]dt + \sigma r(t)dW(t). \tag{3.30}$$

Since $r(t)$ is the exponential of a Gaussian process, it is positive and lognormally distributed. Unlike Dothan, it is mean-reverting toward a positive long-run level:

$$\lim_{t\to\infty} \mathbb{E}[r(t) \mid \mathcal{F}_s] = \exp\left(\frac{\theta}{a} + \frac{\sigma^2}{4a}\right). \tag{9}$$

The variance also converges to a finite value.

The cost is tractability. EV is not affine, does not give explicit formulas for pure discount bonds or bond options, and still shares the lognormal bank-account explosion issue.

Summary

- Dothan fixes the negative-rate problem by making $r(t)$ lognormal, but it loses meaningful mean reversion and has poor pricing tractability.
- CIR uses square-root volatility to keep rates positive while preserving mean reversion and affine bond pricing.
- Affine term-structure models are models where the short rate generates bond prices of the form $A(t,T)e^{-B(t,T)r(t)}$.
- Exponential Vasicek makes $\log r(t)$ Vasicek-like, giving positive mean-reverting rates but losing affine tractability.

Next lecture: Hull-White extended Vasicek and exact fitting of the initial term structure.