Interest Rate Models: Theory and Practice

Problem Set 7

Part A

Part A concerns the mechanics of the Dothan, CIR, and Exponential-Vasicek short-rate models.

Problem A-1. The Dothan model

Under the risk-neutral measure, the Dothan model is

$$dr(t) = ar(t)dt + \sigma r(t)dW(t), \qquad r(0) = r_0.$$

(a) Derive the explicit solution

$$r(t) = r(s)\exp\left\{\left(a - \frac{1}{2}\sigma^2\right)(t-s) + \sigma[W(t) - W(s)]\right\}.$$

(b) Compute $\mathbb{E}[r(t) \mid \mathcal{F}_s]$ and $\mathrm{Var}[r(t) \mid \mathcal{F}_s]$.

(c) Explain why the model keeps the short rate positive.

(d) Explain why the Dothan model does not give the same kind of mean reversion as Vasicek.

Problem A-2. Lognormal short rates and bank-account explosion

Suppose $r(t)$ is lognormally distributed over a short interval.

(a) Explain why approximating

$$\int_0^{\Delta t} r(s)ds \approx \frac{\Delta t}{2}[r(0) + r(\Delta t)]$$

leads to an expectation involving $\mathbb{E}\{\exp(\exp(Y))\}$ for a normal random variable Y.

(b) Explain why this creates a bank-account explosion problem.

(c) Explain why this problem is attached to lognormal short-rate models rather than specifically to Dothan alone.

(d) Explain why this is a serious warning for using the bank account as a numeraire.

Problem A-3. The CIR model

The CIR model is

$$dr(t) = k[\theta - r(t)]dt + \sigma\sqrt{r(t)}dW(t), \qquad r(0) = r_0,$$

where $k, \theta, \sigma, r_0 > 0$.

(a) Interpret the roles of k, θ, and $\sigma\sqrt{r(t)}$.

(b) State the positivity condition and explain its meaning.

(c) Compute or state the conditional mean

$$\mathbb{E}[r(t) \mid \mathcal{F}_s].$$

(d) Compare the mean-reversion structure of CIR with that of Vasicek.

Problem A-4. Exponential Vasicek

Let

$$dy(t) = [\theta - ay(t)]dt + \sigma dW(t), \qquad r(t) = e^{y(t)}.$$

(a) Use Ito's formula to derive the dynamics of $r(t)$.

(b) Explain why $r(t)$ is positive.

(c) Explain why this model improves on Dothan's mean-reversion behavior.

(d) Explain why the model still shares the lognormal short-rate warning.

Part B

Part B concerns affine term structures, bond-pricing tractability, and model comparison.

Problem B-1. Affine bond prices and affine spot rates

Suppose the zero-coupon bond price has the form

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}.$$

(a) Take logarithms and show that the log bond price is affine in $r(t)$.

(b) Use the continuously compounded spot-rate definition to show that

$$R(t,T) = \alpha(t,T) + \beta(t,T)r(t).$$

(c) Identify $\alpha(t,T)$ and $\beta(t,T)$.

(d) Explain why this affine structure is useful for term-structure modeling.

Problem B-2. Affine-coefficient condition

Consider a general short-rate model

$$dr(t) = b(t, r(t))dt + \sigma(t, r(t))dW(t).$$

(a) State the affine-coefficient condition on $b(t,x)$ and $\sigma^2(t,x)$.

(b) Show how Vasicek fits this condition.

(c) Show how CIR fits this condition.

(d) Explain why Dothan and Exponential Vasicek do not fit the same affine term-structure class.

Problem B-3. Forward-rate volatility in affine models

In an affine model,

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}.$$

(a) Starting from $P(t,T) = A(t,T)e^{-B(t,T)r(t)}$, derive the expression for $f(t,T)$.

(b) Show that the instantaneous absolute volatility of $f(t,T)$ is

$$\sigma_f(t,T) = \frac{\partial B(t,T)}{\partial T}\sigma(t, r(t)).$$

(c) Explain why this volatility is deterministic in Vasicek but state-dependent in CIR.

(d) Explain why state-dependent forward-rate volatility makes CIR less algebraically simple than Vasicek.

Problem B-4. Model comparison

Compare Vasicek, Dothan, CIR, and Exponential Vasicek along the following dimensions:

(a) positivity of the short rate;

(b) mean reversion;

(c) affine bond-pricing tractability;

(d) availability of clean bond-option formulas;

(e) the main strength and main weakness of each model.

Your answer should be precise rather than long. The goal is to explain why no classical one-factor model is best in every dimension.