Interest Rate Models: Theory and Practice

Solutions — Problem Set 7

Part A

Part A concerns the mechanics of the Dothan, CIR, and Exponential-Vasicek short-rate models.

Problem A-1. The Dothan model

Under the risk-neutral measure, the Dothan model is

$$dr(t) = ar(t)dt + \sigma r(t)dW(t), \qquad r(0) = r_0.$$

(a) Derive the explicit solution.

(b) Compute $\mathbb{E}[r(t) \mid \mathcal{F}_s]$ and $\mathrm{Var}[r(t) \mid \mathcal{F}_s]$.

(c) Explain why the model keeps the short rate positive.

(d) Explain why the Dothan model does not give the same kind of mean reversion as Vasicek.

Solution. (a) Since the diffusion coefficient is proportional to $r(t)$, apply Ito's formula to $\ln r(t)$. We have

$$d\ln r(t) = \frac{1}{r(t)}dr(t) - \frac{1}{2r(t)^2}(dr(t))^2.$$

Using

$$dr(t) = ar(t)dt + \sigma r(t)dW(t), \qquad (dr(t))^2 = \sigma^2 r(t)^2 dt,$$

we obtain

$$d\ln r(t) = \left(a - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t).$$

Integrating from s to t gives

$$\ln r(t) - \ln r(s) = \left(a - \frac{1}{2}\sigma^2\right)(t-s) + \sigma[W(t) - W(s)].$$

Exponentiating,

$$r(t) = r(s)\exp\left\{\left(a - \frac{1}{2}\sigma^2\right)(t-s) + \sigma[W(t) - W(s)]\right\}.$$

(b) Conditional on $\mathcal{F}_s$, the increment $W(t) - W(s)$ is normal with mean 0 and variance $t-s$. Therefore $r(t)$ is conditionally lognormal. The standard lognormal moment formulas give

$$\mathbb{E}[r(t) \mid \mathcal{F}_s] = r(s)e^{a(t-s)},$$

and

$$\mathrm{Var}[r(t) \mid \mathcal{F}_s] = r(s)^2 e^{2a(t-s)}\left(e^{\sigma^2(t-s)} - 1\right).$$

(c) The solution is the initial short rate multiplied by an exponential. If $r(s) > 0$, then the exponential term is strictly positive for every realization of the Brownian increment. Hence $r(t) > 0$ for all later times in the model.

(d) Vasicek pulls the short rate toward a positive long-run level θ. Dothan does not have such a drift. Its conditional mean is

$$r(s)e^{a(t-s)}.$$

If $a < 0$, the mean tends to zero. If $a = 0$, the mean is constant. If $a > 0$, the mean grows. Thus Dothan fixes positivity, but it does not give economically meaningful mean reversion toward a positive level.

Problem A-2. Lognormal short rates and bank-account explosion

Suppose $r(t)$ is lognormally distributed over a short interval.

(a) Explain why approximating the short-rate integral leads to an expectation involving $\mathbb{E}\{\exp(\exp(Y))\}$.

(b) Explain why this creates a bank-account explosion problem.

(c) Explain why this problem is attached to lognormal short-rate models rather than specifically to Dothan alone.

(d) Explain why this is a serious warning for using the bank account as a numeraire.

Solution. (a) The bank account over a small interval satisfies

$$B(\Delta t) = \exp\left(\int_0^{\Delta t} r(s)ds\right).$$

For small Δt, approximate the integral by the trapezoidal rule:

$$\int_0^{\Delta t} r(s)ds \approx \frac{\Delta t}{2}[r(0) + r(\Delta t)].$$

If $r(\Delta t)$ is lognormal, then $r(\Delta t) = \exp(Y)$ for a normal random variable Y up to deterministic scaling. Hence the expectation of the bank account contains a term of the form

$$\mathbb{E}\{\exp(c\exp(Y))\},$$

for a positive constant c.

(b) The problem is that this expectation may be infinite. Intuitively, the lognormal right tail is heavy enough that exponentiating the lognormal variable creates an expectation too large to be finite. Thus the model can imply that one unit invested in the bank account has infinite expected value over an arbitrarily short interval.

(c) This is not only a Dothan problem. It is a problem for models in which the continuously compounded short rate itself is lognormally distributed. Dothan, Exponential Vasicek, Black-Karasinski, and related lognormal short-rate models all face the same structural warning.

(d) The bank account is the basic numeraire for risk-neutral pricing. If the expected value of the bank account can explode, then the foundational compounding object of the model behaves badly. Even if trees or finite-state approximations make the issue less visible in practice, it is still a serious theoretical warning about the continuous-time model.

Problem A-3. The CIR model

The CIR model is

$$dr(t) = k[\theta - r(t)]dt + \sigma\sqrt{r(t)}dW(t), \qquad r(0) = r_0.$$

(a) Interpret the roles of k, θ, and $\sigma\sqrt{r(t)}$.

(b) State the positivity condition and explain its meaning.

(c) Compute or state the conditional mean.

(d) Compare the mean-reversion structure of CIR with that of Vasicek.

Solution. (a) The drift term is

$$k[\theta - r(t)].$$

The parameter k is the speed of mean reversion. The parameter θ is the long-run level. If $r(t) < \theta$, the drift is positive; if $r(t) > \theta$, the drift is negative. The diffusion coefficient is

$$\sigma\sqrt{r(t)}.$$

Thus volatility increases with the level of rates and shrinks as rates approach zero.

(b) The standard positivity condition is

$$2k\theta > \sigma^2.$$

This condition makes the origin inaccessible to the process. Economically, the pull away from zero is strong enough relative to volatility that the short rate remains positive.

(c) The conditional mean is

$$\mathbb{E}[r(t) \mid \mathcal{F}_s] = r(s)e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}).$$

This is the same fading-old-rate plus long-run-mean-pull structure that appeared in Vasicek.

(d) The mean-reversion structure of CIR is very similar to Vasicek at the level of the conditional mean. In both models, the expected short rate moves toward θ at speed k. The difference is in the volatility. Vasicek has constant absolute volatility σ, while CIR has state-dependent volatility $\sigma\sqrt{r(t)}$. This is what helps CIR preserve positivity.

Problem A-4. Exponential Vasicek

Let

$$dy(t) = [\theta - ay(t)]dt + \sigma dW(t), \qquad r(t) = e^{y(t)}.$$

(a) Use Ito's formula to derive the dynamics of $r(t)$.

(b) Explain why $r(t)$ is positive.

(c) Explain why this model improves on Dothan's mean-reversion behavior.

(d) Explain why the model still shares the lognormal short-rate warning.

Solution. (a) Since $r(t) = e^{y(t)}$, Ito's formula gives

$$dr(t) = e^{y(t)}dy(t) + \frac{1}{2}e^{y(t)}(dy(t))^2.$$

Because

$$dy(t) = [\theta - ay(t)]dt + \sigma dW(t), \qquad (dy(t))^2 = \sigma^2 dt,$$

we get

$$dr(t) = e^{y(t)}\left([\theta - ay(t)]dt + \sigma dW(t)\right) + \frac{1}{2}e^{y(t)}\sigma^2 dt.$$

Using $e^{y(t)} = r(t)$ and $y(t) = \ln r(t)$,

$$dr(t) = r(t)\left[\theta + \frac{\sigma^2}{2} - a\ln r(t)\right]dt + \sigma r(t)dW(t).$$

(b) Since $r(t) = e^{y(t)}$, it is strictly positive for every real value of $y(t)$.

(c) Dothan makes $r(t)$ itself a geometric Brownian motion, so its mean either grows, stays flat, or tends to zero depending on a. Exponential Vasicek instead makes $\ln r(t)$ an Ornstein-Uhlenbeck process. Since

$y(t)$ is mean-reverting, $r(t) = e^{y(t)}$ becomes positive and mean-reverting in distribution. In particular, the limiting mean is positive rather than forced to zero.

(d) The model still makes the short rate lognormal at each date. Therefore expectations involving the exponential of the integral of r can still create bank-account explosion problems. EV improves the mean-reversion behavior relative to Dothan, but it does not remove the lognormal short-rate warning.

Part B

Part B concerns affine term structures, bond-pricing tractability, and model comparison.

Problem B-1. Affine bond prices and affine spot rates

Suppose

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}.$$

(a) Take logarithms and show that the log bond price is affine in $r(t)$.

(b) Use the continuously compounded spot-rate definition to show that $R(t,T) = \alpha(t,T) + \beta(t,T)r(t)$.

(c) Identify $\alpha(t,T)$ and $\beta(t,T)$.

(d) Explain why this affine structure is useful.

Solution. (a) Taking logarithms gives

$$\ln P(t,T) = \ln A(t,T) - B(t,T)r(t).$$

This is affine in $r(t)$ because it is a deterministic constant plus a deterministic coefficient times the state variable.

(b) The continuously compounded spot rate is

$$R(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

Substituting the affine log bond price gives

$$R(t,T) = -\frac{\ln A(t,T) - B(t,T)r(t)}{T-t}.$$

Therefore

$$R(t,T) = -\frac{\ln A(t,T)}{T-t} + \frac{B(t,T)}{T-t}r(t).$$

(c) Hence

$$\alpha(t,T) = -\frac{\ln A(t,T)}{T-t}, \qquad \beta(t,T) = \frac{B(t,T)}{T-t}.$$

(d) The affine structure is useful because it turns the entire zero-coupon curve into a simple function of the state variable $r(t)$. Once A and B are known, the model can produce bond prices and spot rates directly from the current short rate. This is why Vasicek and CIR are analytically important despite their limitations.

Problem B-2. Affine-coefficient condition

Consider

$$dr(t) = b(t, r(t))dt + \sigma(t, r(t))dW(t).$$

(a) State the affine-coefficient condition.

(b) Show how Vasicek fits this condition.

(c) Show how CIR fits this condition.

(d) Explain why Dothan and Exponential Vasicek do not fit the same affine term-structure class.

Solution. (a) The affine-coefficient condition is that both the drift and the instantaneous variance are affine functions of the state variable:

$$b(t,x) = \lambda(t)x + \eta(t), \qquad \sigma^2(t,x) = \gamma(t)x + \delta(t),$$

for deterministic functions $\lambda, \eta, \gamma, \delta$.

(b) In Vasicek,

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t).$$

Thus

$$b(t,x) = -kx + k\theta, \qquad \sigma^2(t,x) = \sigma^2.$$

This corresponds to

$$\lambda = -k, \quad \eta = k\theta, \quad \gamma = 0, \quad \delta = \sigma^2.$$

So Vasicek satisfies the affine-coefficient condition.

(c) In CIR,

$$dr(t) = k[\theta - r(t)]dt + \sigma\sqrt{r(t)}dW(t).$$

Thus

$$b(t,x) = -kx + k\theta, \qquad \sigma^2(t,x) = \sigma^2 x.$$

This corresponds to

$$\lambda = -k, \quad \eta = k\theta, \quad \gamma = \sigma^2, \quad \delta = 0.$$

So CIR also satisfies the affine-coefficient condition.

(d) In Dothan,

$$dr(t) = ar(t)dt + \sigma r(t)dW(t),$$

so

$$\sigma^2(t,x) = \sigma^2 x^2,$$

which is quadratic, not affine. In Exponential Vasicek,

$$dr(t) = r(t)\left[\theta + \frac{\sigma^2}{2} - a\ln r(t)\right]dt + \sigma r(t)dW(t),$$

so both the drift contains $x \ln x$ and the variance is $\sigma^2 x^2$. These are not affine in x. Therefore neither Dothan nor EV belongs to the same affine term-structure class.

Problem B-3. Forward-rate volatility in affine models

In an affine model,

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}.$$

(a) Derive the expression for $f(t,T)$.

(b) Show that $\sigma_f(t,T) = \frac{\partial B(t,T)}{\partial T}\sigma(t, r(t))$.

(c) Explain why this volatility is deterministic in Vasicek but state-dependent in CIR.

(d) Explain why state-dependent forward-rate volatility makes CIR less algebraically simple than Vasicek.

Solution. (a) Since

$$\ln P(t,T) = \ln A(t,T) - B(t,T)r(t),$$

we have

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} = -\frac{\partial \ln A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T} r(t).$$

(b) The only stochastic term in this expression is the term involving $r(t)$. If

$$dr(t) = b(t, r(t))dt + \sigma(t, r(t))dW(t),$$

then applying Ito's formula to the expression for $f(t,T)$ gives diffusion coefficient

$$\frac{\partial B(t,T)}{\partial T}\sigma(t, r(t)).$$

Thus

$$\sigma_f(t,T) = \frac{\partial B(t,T)}{\partial T}\sigma(t, r(t)).$$

(c) In Vasicek, the short-rate diffusion coefficient is constant:

$$\sigma(t, r(t)) = \sigma.$$

Therefore forward-rate volatility is deterministic once $B(t,T)$ is known. In CIR,

$$\sigma(t, r(t)) = \sigma\sqrt{r(t)},$$

so the forward-rate volatility depends on the current level of the short rate.

(d) State-dependent volatility means that the distributional and pricing calculations depend on the current state in a nonlinear way. Vasicek remains Gaussian with deterministic volatility loadings, while CIR involves noncentral chi-squared distributions and more complicated option-pricing expressions. CIR is still tractable, but it is not as algebraically clean as Vasicek.

Problem B-4. Model comparison

Compare Vasicek, Dothan, CIR, and Exponential Vasicek along positivity, mean reversion, affine tractability, bond-option formulas, and main strengths and weaknesses.

Solution. One possible comparison is as follows.

Model	Positive?	Mean-reverting?	Affine?	Clean bond-option formula?
Vasicek	No	Yes	Yes	Yes
Dothan	Yes	Only weakly	No	No
CIR	Yes, under condition	Yes	Yes	Yes, but more complex
EV	Yes	Yes	No	No

Vasicek is the cleanest Gaussian model: it is mean-reverting, affine, and analytically tractable, but it permits negative rates. Dothan fixes positivity by making the short rate lognormal, but it loses meaningful mean reversion, is not affine, and has poor derivative-pricing tractability. CIR is the strongest classical compromise in this group: it keeps rates positive under the Feller condition, preserves mean reversion, and remains affine. Its cost is that the distribution and option formulas are more complicated than in Vasicek. Exponential Vasicek gives positive and mean-reverting rates by making the log short rate Vasicek-like, but it is not affine and does not provide explicit pure-discount bond or bond-option formulas.

Thus no classical one-factor model dominates in every dimension. Each model chooses a different tradeoff among realism of rate behavior, analytical tractability, and pricing convenience.