Interest Rate Models — Lecture 8 Notes

Section 3.3: The Hull–White Extended Vasicek Model

1 The Hull–White extension of Vasicek

The classical Vasicek model is analytically tractable, but its constant parameters generally prevent it from fitting an arbitrary initial market term structure. The Hull–White extended Vasicek model keeps the Gaussian short-rate structure, but replaces the constant drift level with an exogenous, time-dependent deterministic function $\vartheta(t)$. Under the risk-neutral pricing measure $\mathbb{Q}$, the short rate satisfies

$$dr(t) = \big[\vartheta(t) - ar(t)\big]dt + \sigma dW(t),$$

where $a > 0$ is the constant mean-reversion speed and $\sigma > 0$ is the instantaneous volatility parameter. The model remains strictly Gaussian. Multiplying the SDE by the integrating factor e^{at} and integrating over the finite horizon $[s, t]$ yields the explicit state solution:

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\vartheta(u)\,du + \sigma \int_s^t e^{-a(t-u)}\,dW(u).$$

Let $\mathcal{F}_s$ denote the filtration representing all market information accumulated up to time s. Because the stochastic integrand is completely deterministic, the path-dependent integral defines a Gaussian process whose conditional distribution is fully characterized by its first two moments:

$$\mathbb{E}[r(t) \mid \mathcal{F}_s] = r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\vartheta(u)\,du, \qquad \operatorname{Var}(r(t) \mid \mathcal{F}_s) = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right).$$

For analytical convenience, the short rate is deconstructed into a random tracking error and a smooth baseline via $r(t) = x(t) + \alpha(t)$, where x represents the zero-mean Ornstein–Uhlenbeck component satisfying

$$dx(t) = -ax(t)dt + \sigma dW(t), \qquad x(0) = 0.$$

The deterministic shift function $\alpha(t)$ is isolated and chosen directly to fit the initial market term structure.

2 Exact fitting of the initial term structure

Let $P^M(0,T)$ denote the market zero-coupon bond price observed at time 0, and let $f^M(0,t) = -\frac{\partial \log P^M(0,t)}{\partial t}$ be the market instantaneous forward rate. Hull–White chooses $\vartheta(t)$ so that the model reproduces the initial market curve exactly. The required continuous analytical drift function is

$$\vartheta(t) = \frac{\partial f^M(0,t)}{\partial t} + af^M(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

Equivalently, substituting this calibration parameter back into the integrated system reveals that the deterministic shift mapping $r(t) = x(t) + \alpha(t)$ evaluates explicitly to:

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2.$$

Thus, the model absorbs the exact term structure of the market while preserving the tractability of the Vasicek framework. The structural cost of this continuous Gaussian tractability is that the short rate can become negative with positive probability. Because $r(t)$ retains infinite tails, the absolute probability of crossing the zero boundary is:

$$\mathbb{Q}\{r(t) < 0\} = \Phi\left(\frac{-\alpha(t)}{\sqrt{\frac{\sigma^2}{2a}(1 - e^{-2at})}}\right),$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function.

3 Affine bond pricing

The Hull–White model belongs to the affine class, meaning log-bond prices are linear functions of the short rate. The price of a zero-coupon bond maturing at T is given by $P(t,T) = A(t,T)e^{-B(t,T)r(t)}$, where the state variable sensitivity $B(t,T)$ and the deterministic calibration coefficient $A(t,T)$ are defined as:

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}, \quad A(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left\{B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2at})B(t,T)^2\right\}.$$

The exponential multiplier inside $A(t,T)$ functions as a convexity adjustment, stripping out the geometric distortion caused by the non-linear curvature of the price-yield relationship. The framework ensures that the initial curve enters directly through observed market bond prices and forward rates, while remaining one-dimensional through $r(t)$.

4 Zero-coupon bond options

Consider a European call option expiring at T on a zero-coupon bond maturing at $S > T$ with strike X. The time-t value is $\mathrm{ZBC}(t,T,S,X) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}(P(T,S) - X)^+ \mid \mathcal{F}_t\right]$. We execute a change of numeraire from the bank account to the T-forward measure $\mathbb{Q}^T$ associated with the numeraire bond $P(\cdot,T)$, transforming the expectation into:

$$\mathrm{ZBC}(t,T,S,X) = P(t,T)\,\mathbb{E}^T\left[(P(T,S) - X)^+ \mid \mathcal{F}_t\right].$$

Under $\mathbb{Q}^T$, the underlying bond price $P(T,S)$ is lognormally distributed, yielding an analytical, closed-form Black-type pricing formula. Symmetrically applying put-call parity resolves the corresponding European put option:

$$\begin{aligned}\mathrm{ZBC}(t,T,S,X) &= P(t,S)\Phi(h) - XP(t,T)\Phi(h - \sigma_p)\\ \mathrm{ZBP}(t,T,S,X) &= XP(t,T)\Phi(-h + \sigma_p) - P(t,S)\Phi(-h)\end{aligned}$$

where the total integrated volatility parameter σ_p and the normalized moneyness threshold Z-score h are defined as:

$$\sigma_p = \sigma\sqrt{\frac{1 - e^{-2a(T-t)}}{2a}}B(T,S), \qquad h = \frac{1}{\sigma_p}\log\left(\frac{P(t,S)}{P(t,T)X}\right) + \frac{\sigma_p}{2}.$$

5 Caps and floors

An institutional interest rate caplet with reset date t_{i-1}, payment date t_i, accrual year fraction τ_i, nominal principal N, and cap strike X pays out the cash flow $N\tau_i(L(t_{i-1}, t_i) - X)^+$ at time t_i. Substituting the definition of simply compounded spot LIBOR, $L(t_{i-1}, t_i) = \frac{1-P(t_{i-1},t_i)}{\tau_i P(t_{i-1},t_i)}$, maps the payoff directly to a put option on a zero-coupon bond:

$$N\tau_i \left(\frac{1 - P(t_{i-1}, t_i)}{\tau_i P(t_{i-1}, t_i)} - X \right)^+ = \frac{N(1 + X\tau_i)}{P(t_{i-1}, t_i)} \left(\frac{1}{1 + X\tau_i} - P(t_{i-1}, t_i) \right)^+ .$$

By summing sequential caplet and floorlet components across the timeline $\mathcal{T}$, caps and floors are priced as portfolios of bond options:

$$\mathrm{Cap}(t, \mathcal{T}, N, X) = N \sum_{i=1}^{n} \left[P(t, t_{i-1})\Phi(-h_i + \sigma_p^i) - (1 + X\tau_i)P(t, t_i)\Phi(-h_i) \right] ,$$

$$\mathrm{Flr}(t, \mathcal{T}, N, X) = N \sum_{i=1}^{n} \left[(1 + X\tau_i)P(t, t_i)\Phi(h_i) - P(t, t_{i-1})\Phi(h_i - \sigma_p^i) \right] ,$$

where σ_p^i and h_i are evaluated across the localized steps $T = t_{i-1}$ and $S = t_i$. Once zero-coupon bond options are available in closed form, macro contract portfolios are obtained directly by summation.

6 Coupon-bond options and Jamshidian's decomposition

Options on coupon-bearing bonds are complex because a single strike price X applies directly to a consolidated sum of multiple cash flows $\sum_{i=1}^{n} c_i P(T, t_i)$. Because Hull–White is a one-factor model, every point along the yield curve moves in monotonic lockstep with $r(t)$, meaning each component bond price $P(T, t_i)$ is a strictly decreasing function of the single state variable $r(T)$.

Jamshidian's decomposition leverages this monotonicity. We execute a one-dimensional numerical search to isolate the unique critical short-rate state r^* at which the total market value of the underlying coupon bond exactly equals par:

$$\sum_{i=1}^{n} c_i A(T, t_i) e^{-B(T,t_i)r^*} = 1.$$

This critical short rate defines individual zero-coupon bond-option strike allocations: $X_i = A(T, t_i)e^{-B(T,t_i)r^*}$. This structural property allows a multi-cashflow payer swaption (PS) or coupon bond option to be decomposed into a simple sum of one-period bond options:

$$\mathrm{PS}(t, T, \mathcal{T}, N, X) = N \sum_{i=1}^{n} c_i \,\mathrm{ZBP}(t, T, t_i, X_i).$$

7 Trinomial tree construction

For path-dependent and early-exercise derivatives, continuous calculus is mapped to a discrete recombining tree. The tree geometry is built first for the zero-mean process $dx(t) = -ax(t)dt + \sigma dW(t)$, and the real-world calibration shift α_i is added column-by-column later via $r_{i,j} = x_{i,j} + \alpha_i$. Let the time grid be $0 = t_0 < t_1 < \cdots < t_n$ with local step size $\Delta t_i = t_{i+1} - t_i$. At node (i, j), the state value is $x_{i,j} = j\Delta x_i$. The continuous local target mean and variance required to match continuous physics over one step are:

$$M_{i,j} = x_{i,j} e^{-a\Delta t_i}, \qquad V_i^2 = \frac{\sigma^2}{2a} \left(1 - e^{-2a\Delta t_i} \right) .$$

To guarantee absolute numerical grid stability, the vertical space interval separating adjacent nodes is fixed as:

$$\Delta x_i = V_{i-1}\sqrt{3}.$$

From node (i, j), the lattice branches into exactly three nodes at the next time step: $k+1$, k, and $k-1$. Under the pressure of mean reversion, the tree dynamically adapts by tilting its branching fork using a rounding mechanism to center branches on the rail closest to the local target mean:

$$k = \mathrm{round}\left(\frac{M_{i,j}}{\Delta x_{i+1}}\right).$$

Let $\eta_{j,k} = M_{i,j} - x_{i+1,k}$ define the spatial misalignment rounding error. The transition probabilities are calibrated to match the continuous local moments exactly:

$$p_u = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} + \frac{\eta_{j,k}}{2\sqrt{3}V_i}, \qquad p_m = \frac{2}{3} - \frac{\eta_{j,k}^2}{3V_i^2}, \qquad p_d = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} - \frac{\eta_{j,k}}{2\sqrt{3}V_i}.$$

8 Arrow–Debreu prices and discrete calibration

Let $Q_{i,j}$ denote the Arrow–Debreu price of reaching node (i, j), representing the time-zero value of a claim paying one unit if the tree reaches that specific node and zero otherwise. These prices are computed inductively forward through the lattice:

$$Q_{i+1,j} = \sum_h Q_{i,h}\, q(h, j) \exp\left(-(\alpha_i + h\Delta x_i)\Delta t_i\right),$$

where $q(h, j)$ is the transition probability from node (i, h) to node $(i+1, j)$. At each step, the discrete vertical lift α_i is isolated and solved by forcing the total discounted value of the node tickets to exactly match the observed market bond price $P^M(0, t_{i+1})$:

$$\alpha_i = \frac{1}{\Delta t_i} \log\left(\frac{\sum_{j=\underline{j}_i}^{\bar{j}_i} Q_{i,j} \exp(-j\Delta x_i \Delta t_i)}{P^M(0, t_{i+1})}\right).$$

The tree cleanly separates two tasks: the zero-mean process x provides the stable, recombining stochastic geometry, while the deterministic shifts α_i force exact calibration to the observed initial market curve.

Summary

- Hull–White extends Vasicek by replacing the constant drift level with a deterministic function $\vartheta(t)$.
- The model remains Gaussian, one-factor, and analytically tractable, with an affine pricing structure.
- The deterministic shift is chosen to fit the initial market zero-coupon curve exactly.
- Zero-coupon bond options have closed-form Black-type prices; caps and floors are portfolios of these options.
- Jamshidian's decomposition reduces multi-cashflow options and swaptions to simple portfolios of one-period bond options.
- Trinomial trees provide numerical solutions for early-exercise products, using Arrow–Debreu prices to enforce exact calibration.