Interest Rate Models — Problem Set 8

Section 3.3: Continuous Dynamics, Exact Fitting, and Trinomial Tree Calibration

Goal: This problem set evaluates your mathematical and numerical mastery of the Hull–White extended Vasicek model. You will verify continuous dynamics, change measures, decompose complex coupon options, and calibrate a discrete recombining lattice using forward induction.

Part A: Analytical Dynamics, Calibration, and Closed-Form Options

Problem A-1. SDE Integration and Initial Drift Extraction

Under the risk-neutral measure $\mathbb{Q}$, the Hull–White short-rate dynamics satisfy the stochastic differential equation:

$$dr(t) = \big[\vartheta(t) - ar(t)\big]dt + \sigma dW(t), \qquad a, \sigma > 0.$$

(a) Define the transformation $f(t, r) = r(t)e^{at}$. Apply Itô's lemma to verify that the integrated analytical solution for the short rate over the finite horizon $[s, t]$ is given by:

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\vartheta(u)\,du + \sigma \int_s^t e^{-a(t-u)}\,dW(u).$$

(b) Using the properties of stochastic integrals with deterministic integrands, prove that the conditional mean and conditional variance are:

$$\mathbb{E}[r(t) \mid \mathcal{F}_s] = r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\vartheta(u)\,du, \qquad \mathrm{Var}(r(t) \mid \mathcal{F}_s) = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right).$$

(c) Let $f^M(0, t) = -\frac{\partial \log P^M(0,t)}{\partial t}$ denote the market instantaneous forward rate observed today. Show that to perfectly reproduce the initial term structure, the continuous deterministic drift baseline must satisfy:

$$\vartheta(t) = \frac{\partial f^M(0, t)}{\partial t} + af^M(0, t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

(d) Given that the short rate is normally distributed, write an explicit expression for the risk-neutral probability that the rate falls below zero at a future date t, $\mathbb{Q}\{r(t) < 0\}$. Explain why this structural property is considered the main limitation of the model.

Problem A-2. The T-Forward Measure and Zero-Coupon Bond Options

Consider a European call option expiring at T with strike X on a zero-coupon bond maturing at $S > T$. The time-t price of this contract can be stated under the risk-neutral measure as:

$$\mathrm{ZBC}(t, T, S, X) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}(P(T, S) - X)^+ \mid \mathcal{F}_t\right].$$

(a) Define the Radon–Nikodym derivative $\frac{d\mathbb{Q}^T}{d\mathbb{Q}}$ that transforms the pricing environment to the T-forward measure using the zero-coupon bond $P(\cdot, T)$ as the numeraire. (b) Show that under this change of numeraire, the short-rate tracking error component $dx(t) = -ax(t)dt + \sigma dW(t)$ absorbs a target drift distortion, satisfying:

$$dx(t) = \left[-B(t, T)\sigma^2 - ax(t)\right]dt + \sigma dW^T(t), \qquad \text{where } B(t, T) = \frac{1 - e^{-a(T-t)}}{a}.$$

(c) Derive the closed-form Black-type formula for $\text{ZBC}(t, T, S, X)$ using this forward measure. Explain the exact financial and statistical role played by the integrated volatility parameter:

$$\sigma_p = \sigma \sqrt{\frac{1 - e^{-2a(T-t)}}{2a}} B(T, S).$$

(d) State the corresponding closed-form formula for a zero-coupon bond put option (ZBP) by applying put-call parity.

Problem A-3. Jamshidian's Decomposition for Coupon Options

A European option written on a coupon-bearing bond with maturity T possesses an underlying payment schedule delivering cash flows c_i at sequential dates $t_i > T$ $(i = 1, \ldots, n)$. The aggregate strike price is X. (a) Explain why pricing this option is mathematically difficult when the underlying asset is treated as a consolidated multi-cashflow portfolio. (b) State the monotonicity property of a one-factor short-rate model. Explain why every individual zero-coupon bond component $P(T, t_i)$ must move in perfect lockstep with the state variable $r(T)$. (c) Formulate the mathematical conditions for a numerical root-finding algorithm to isolate the unique critical short rate r^* that satisfies:

$$\sum_{i=1}^{n} c_i A(T, t_i) e^{-B(T, t_i) r^*} = X.$$

(d) Prove that once r^* is located, a European put option on this coupon bond can be perfectly decomposed into a simple summation of individual zero-coupon bond puts:

$$\text{PS}(t, T, \mathcal{T}, N, X) = N \sum_{i=1}^{n} c_i \, \text{ZBP}(t, T, t_i, X_i), \qquad \text{where } X_i = A(T, t_i) e^{-B(T, t_i) r^*}.$$

Part B: Discrete Numerical Lattices and Arrow–Debreu Calibration

Problem B-1. Mean Reversion and Dynamic Branch Selection

To build a stable numerical grid, we first construct a recombining trinomial tree for the zero-mean process $dx(t) = -ax(t)dt + \sigma dW(t)$ on a discretized timeline $\Delta t_i = t_{i+1} - t_i$, where node coordinates are defined via space indexes $x_{i,j} = j\Delta x_i$. (a) State the conditional mean $M_{i,j}$ and conditional variance V_i^2 over a single horizontal time step. Why is the vertical node separation fixed to the scale $\Delta x_i = V_{i-1}\sqrt{3}$? (b) Suppose a model builder constructs a trinomial tree where the branching paths from node (i, j) always target the straight horizontal rails across time: $j + 1$ (Up), j (Middle), and $j - 1$ (Down). Explain mathematically why this naive lattice layout fails and encounters probability breakdown under mean reversion. (c) Define the dynamic branching row adjustment coordinate:

$$k = \text{round}\left(\frac{M_{i,j}}{\Delta x_{i+1}}\right).$$

Explain how this rounding mechanism prevents probability breakdown by tracking the continuous drift expectations of the process. (d) Sketch or describe the three distinct branching topologies that can emerge at a given node depending on whether the process is inside the typical variance band, floating high above the long-term mean, or falling far below it.

Problem B-2. Transition Probability Constraints

Let $\eta_{j,k} = M_{i,j} - x_{i+1,k}$ represent the spatial misalignment error between our rounded node coordinate target and the true continuous expectation. (a) Set up the three independent algebraic moment-matching equations (total probability, expectation preservation, and variance alignment) required to calibrate the local branch probabilities p_u, p_m, and p_d. (b) Solve this linear system to verify that the calibrated transition probabilities are given by:

$$p_u = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} + \frac{\eta_{j,k}}{2\sqrt{3}V_i}, \qquad p_m = \frac{2}{3} - \frac{\eta_{j,k}^2}{3V_i^2}, \qquad p_d = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} - \frac{\eta_{j,k}}{2\sqrt{3}V_i}.$$

(c) Prove that for these expressions to remain strictly bounded as valid probabilities ($0 \leq p_u, p_m, p_d \leq 1$), the maximum spatial alignment error must satisfy $|\eta_{j,k}| \leq \frac{V_i}{\sqrt{3}}$. (d) Show how this constraint validates our choice of the node coordinate rounding function k.

Problem B-3. Arrow–Debreu Forward Induction Loops

Let $Q_{i,j}$ denote the state Arrow–Debreu price of node (i, j), representing the time-0 price of a derivative that pays exactly 1 unit if the short-rate error state matches $x_{i,j}$ at step i, and 0 otherwise. The true market rate mapped to the lattice includes a deterministic column-specific shift: $r_{i,j} = x_{i,j} + \alpha_i$. (a) Write the explicit recursive equation for computing the next column of Arrow–Debreu prices, $Q_{i+1,j}$, using the preceding state space matrix entries $Q_{i,h}$ and transition mappings $q(h, j)$. (b) Explain why computing Arrow–Debreu prices requires an inductive *forward* loop across time steps $i = 0, 1, \ldots, n$, whereas option pricing models require a *backward* recursive loop over the grid. (c) Show that by forcing the tree to match the observed trading screen market zero-coupon bond price $P^M(0, t_{i+1})$ at each step, the required discrete vertical shift evaluates exactly to:

$$\alpha_i = \frac{1}{\Delta t_i} \log\left(\frac{\sum_{j=\underline{j}_i}^{\bar{j}_i} Q_{i,j} \exp(-j\Delta x_i \Delta t_i)}{P^M(0, t_{i+1})}\right).$$

(d) Summarize the structural and computational advantages of this staging method (building a zero-mean tree structure first and then applying vertical adjustments α_i later) compared to re-building a brand new non-recombining asymmetrical tree from scratch.