Interest Rate Models: Theory and Practice

Problem Set 9

Sections 3.4–3.5: CIR extensions and the Black–Karasinski model

Part A

Part A concerns the difficulty of extending CIR with time-dependent coefficients and the basic Black–Karasinski construction.

Problem A-1. Time-dependent CIR extensions

A natural extension of the CIR model is

$$dr(t) = [\vartheta(t) - a(t)r(t)]dt + \sigma(t)\sqrt{r(t)}\,dW(t). \tag{3.51}$$

A simpler version keeps a and σ constant:

$$dr(t) = [\vartheta(t) - ar(t)]dt + \sigma\sqrt{r(t)}\,dW(t). \tag{3.52}$$

(a) Explain why this extension is tempting after studying Hull–White.

(b) State the classical CIR positivity condition and explain why it becomes fragile when the mean level is time-dependent.

(c) Explain why the model is less analytically convenient than Hull–White extended Vasicek.

(d) Why can the explicit dependence on the current short rate $r(t)$ be undesirable for long-dated option pricing?

Problem A-2. The Black–Karasinski short-rate dynamics

The Black–Karasinski model assumes

$$d\ln r(t) = [\theta(t) - a(t)\ln r(t)]dt + \sigma(t)dW(t), \qquad r(0) = r_0 > 0. \tag{3.53}$$

A common simplified version is

$$d\ln r(t) = [\theta(t) - a\ln r(t)]dt + \sigma dW(t). \tag{3.54}$$

(a) Explain why the model keeps the short rate positive.

(b) Use Ito's formula to derive the dynamics of $r(t)$ under the simplified version.

(c) Explain the role of a and σ in the model.

(d) Compare BK with Exponential Vasicek in one or two sentences.

Part B

Part B concerns the lognormal moments and the tree calibration idea.

Problem B-1. Conditional moments and the deterministic log-rate center

For $s \le t$, the solution of the simplified BK log-rate equation implies

$$r(t) = \exp\left(\ln r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \sigma\int_s^t e^{-a(t-u)}dW(u)\right).$$

(a) Explain why $r(t)$ is conditionally lognormal given $\mathcal{F}_s$.

(b) State the conditional first moment $E_s[r(t)]$.

(c) Define

$$\alpha(t) = \ln(r_0)e^{-at} + \int_0^t e^{-a(t-u)}\theta(u)du. \tag{3.55}$$

Explain the meaning of $\alpha(t)$.

(d) Explain why the decomposition $r(t) = e^{\alpha(t)+x(t)}$ is useful for building a tree.

Problem B-2. Black–Karasinski trinomial-tree calibration

In the BK tree, the normal state variable has nodes

$$x_{i,j} = j\Delta x_i,$$

and the node short rate is

$$r_{i,j} = \exp(x_{i,j} + \alpha_i).$$

Let $Q_{i,j}$ denote the time-0 value of a security paying 1 if node (i,j) is reached and 0 otherwise.

(a) Explain why the tree is built for x rather than directly for r.

(b) Write the one-step state-price propagation formula after α_i is known.

(c) Write the equation used to choose α_i so the tree matches the next market bond price $P^M(0, t_{i+1})$.

(d) In words, summarize the BK tree algorithm.