Interest Rate Models: Theory and Practice

Solutions to Problem Set 9

Sections 3.4–3.5: CIR extensions and the Black–Karasinski model

Problem A-1. Time-dependent CIR extensions

A natural extension of the CIR model is

$$dr(t) = [\vartheta(t) - a(t)r(t)]dt + \sigma(t)\sqrt{r(t)}\,dW(t). \tag{3.51}$$

A simpler version keeps a and σ constant:

$$dr(t) = [\vartheta(t) - ar(t)]dt + \sigma\sqrt{r(t)}\,dW(t). \tag{3.52}$$

Solution.

(a) The extension is tempting because Hull–White improved Vasicek by making the drift level time-dependent, allowing the model to fit the initial market curve exactly. The analogous thought is: keep CIR's square-root volatility and positive-rate intuition, but let the coefficients move through time so the model can match the observed term structure.

(b) In the classical constant-parameter CIR model,

$$dr(t) = k[\theta - r(t)]dt + \sigma\sqrt{r(t)}\,dW(t),$$

the usual strict positivity condition is

$$2k\theta > \sigma^2.$$

It says the mean-reversion pull away from zero is strong enough relative to volatility that the origin is not reached. In a time-dependent extension, the mean level is effectively forced to follow the market curve. If the market curve is very low, the fitted time-dependent level can become low as well, weakening the pull away from zero and making positivity less robust.

(c) Hull–White remains Gaussian and keeps many calculations explicit. The time-dependent CIR extension keeps an affine-looking bond-price form,

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

but the Riccati equation for B is explicitly solvable only in special cases. Thus even bond pricing may require numerical work, and option pricing becomes less clean.

(d) Long-dated option prices ideally depend on observable curve quantities such as $P(t,T)$ and $P(t,S)$ rather than on the instantaneous short rate as an extra state input. If the formula keeps explicit dependence on $r(t)$, then pricing and hedging become more sensitive to an unobservable instantaneous quantity.

Problem A-2. The Black–Karasinski short-rate dynamics

The Black–Karasinski model assumes

$$d\ln r(t) = [\theta(t) - a(t)\ln r(t)]dt + \sigma(t)dW(t), \qquad r(0) = r_0 > 0. \tag{3.53}$$

A common simplified version is

$$d\ln r(t) = [\theta(t) - a\ln r(t)]dt + \sigma dW(t). \tag{3.54}$$

Solution.

(a) The model keeps the short rate positive because it models the logarithm of the short rate as the Gaussian state variable. Since

$$r(t) = e^{\ln r(t)},$$

the short rate is strictly positive for every real value of $\ln r(t)$.

(b) Let $y(t) = \ln r(t)$, so $r(t) = e^{y(t)}$. Ito's formula gives

$$dr(t) = e^{y(t)}dy(t) + \frac{1}{2}e^{y(t)}(dy(t))^2.$$

Using

$$dy(t) = [\theta(t) - ay(t)]dt + \sigma dW(t), \qquad (dy(t))^2 = \sigma^2 dt,$$

we get

$$dr(t) = r(t)\left[\theta(t) - a\ln r(t) + \frac{\sigma^2}{2}\right]dt + \sigma r(t)dW(t).$$

Equivalently,

$$dr(t) = r(t)\left[\theta(t) + \frac{\sigma^2}{2} - a\ln r(t)\right]dt + \sigma r(t)dW(t).$$

(c) The parameter a is the speed at which the log-rate is pulled back toward its deterministic center. The parameter σ is the instantaneous volatility of the log-rate. Since $dr(t)$ has diffusion coefficient $\sigma r(t)$, σ is also the instantaneous proportional volatility of the short rate.

(d) BK is like Exponential Vasicek because both models make the log short rate Gaussian and mean-reverting, which guarantees positive rates. The key difference is that BK introduces time-dependent drift functions, especially $\theta(t)$, so the model can be calibrated to the initial term structure through a tree.

Problem B-1. Conditional moments and the deterministic log-rate center

For $s \le t$,

$$r(t) = \exp\left(\ln r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \sigma\int_s^t e^{-a(t-u)}dW(u)\right).$$

Solution.

(a) Conditional on $\mathcal{F}_s$, the first two terms in the exponent are known. The stochastic integral is normally distributed with mean zero and variance

$$\sigma^2\int_s^t e^{-2a(t-u)}du = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right).$$

Thus the exponent is conditionally normal, so $r(t)$ is conditionally lognormal.

(b) If a random variable has the form e^Y with $Y \sim N(m, v)$, then $E[e^Y] = e^{m+v/2}$. Here

$$m = \ln r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du,$$

and

$$v = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right).$$

Therefore

$$E_s[r(t)] = \exp\left(\ln r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \frac{\sigma^2}{4a}\left(1 - e^{-2a(t-s)}\right)\right).$$

(c) The function

$$\alpha(t) = \ln(r_0)e^{-at} + \int_0^t e^{-a(t-u)}\theta(u)du \tag{3.55}$$

is the deterministic center of the log short rate. It is the part of $\ln r(t)$ that would remain if the Brownian shock were removed.

(d) The decomposition

$$r(t) = e^{\alpha(t)+x(t)}$$

separates the model into a deterministic fitting component $\alpha(t)$ and a zero-mean stochastic Gaussian component $x(t)$. This is useful because the recombining tree can be built for the normal variable x, while $\alpha(t)$ is solved time layer by time layer to fit the market discount curve.

Problem B-2. Black–Karasinski trinomial-tree calibration

In the BK tree, the normal state variable has nodes

$$x_{i,j} = j\Delta x_i,$$

and the node short rate is

$$r_{i,j} = \exp(x_{i,j} + \alpha_i).$$

Let $Q_{i,j}$ denote the time-0 value of a security paying 1 if node (i, j) is reached and 0 otherwise.

Solution.

(a) The tree is built for x because x is the Gaussian state variable. A recombining trinomial tree works naturally for a normal mean-reverting process. The short rate itself is then obtained by exponentiating: $r_{i,j} = \exp(x_{i,j} + \alpha_i)$.

(b) Once α_i is known, the state prices are propagated by discounting one step at the node short rate:

$$Q_{i+1,j} = \sum_h Q_{i,h}q(h,j)\exp\left[-\exp(\alpha_i + h\Delta x_i)\Delta t_i\right],$$

where $q(h, j)$ is the transition probability from node (i, h) to node $(i + 1, j)$.

(c) The deterministic shift α_i is chosen so that the tree reproduces the next market zero-coupon bond price:

$$P^M(0, t_{i+1}) - \sum_{j=j_i}^{j^i} Q_{i,j}\exp\left[-\exp(\alpha_i + j\Delta x_i)\Delta t_i\right] = 0.$$

This is a one-dimensional root-finding problem for α_i at each time layer.

(d) The algorithm is: build the normal trinomial tree for x, start with known state prices, solve the current layer's shift α_i so the tree matches $P^M(0, t_{i+1})$, convert each node to a positive short rate by exponentiating, propagate the state prices forward, and repeat for the next layer.